Online Combinatorial Optimization Problems

Mário César San Felice

Instituto de Matemática e Estatística - Universidade de São Paulo

felice@ic.unicamp.br

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Set of inputs and set of solutions
Cost associated with each pair (input, solution)

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As an example, lets take the Steiner Tree Problem

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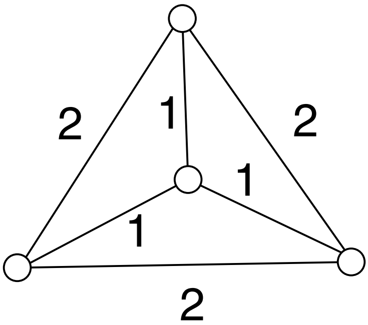
Input: $G = (V, E)$, $d : E \to \mathbb{R}^+$, $D \subseteq V$
Solution: tree T connecting terminal nodes D
Cost: $\sum_{e \in T} d(e)$



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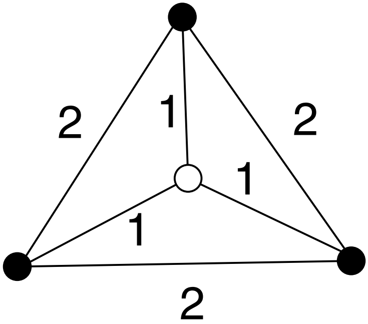

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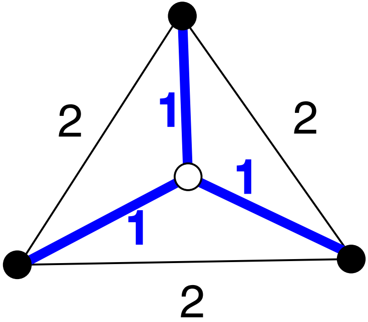

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Input parts arrive one at a time
Each part is served before next one arrives
No decision can be changed in the future

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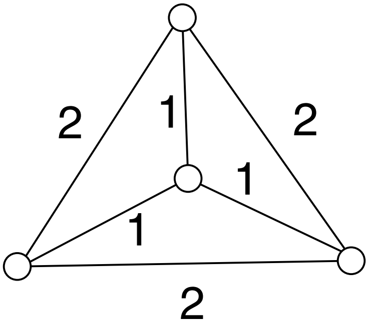
Similar to the Steiner Tree problem
Terminal nodes arrive one at a time
No edge used can be removed in the future



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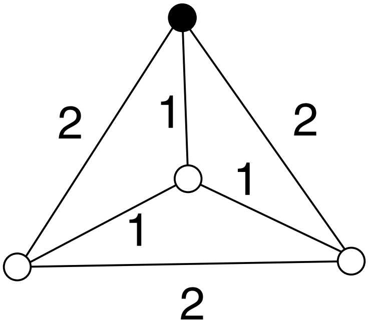

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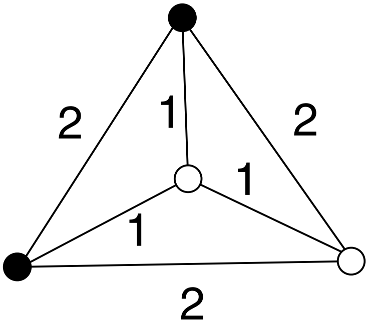

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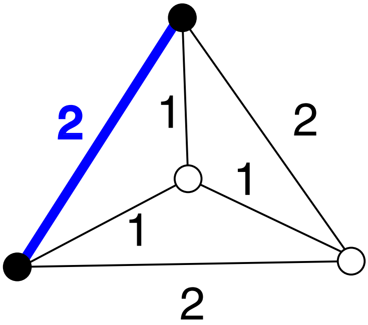

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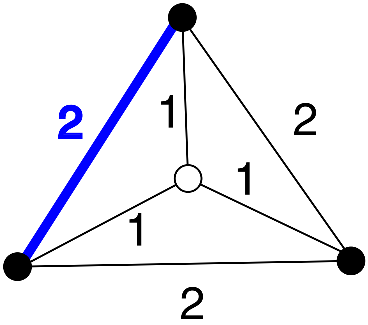

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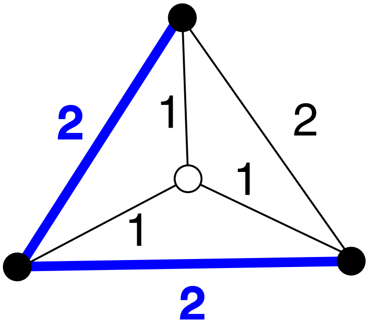

Competitive Analysis

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Worst case analysis technique
For online algorithm ALG
Using offline optimal solution OPT

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As an example, lets take a greedy online algorithm for the Online Steiner Tree problem

Greedy Online Steiner Tree Algorithm

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Algorithm 1: OST Algorithm

Input: (G, d)
$T \leftarrow (\emptyset, \emptyset)$;
while *a new terminal j arrives* **do**
    $T \leftarrow T \cup \{\mathrm{path}(j, V(T))\}$;
end
return T;

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This algorithm is $\mathrm{O}(\log n)$-competitive [Imase and Waxman 1991]

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A $\Omega(\log n)$ lower bound is known [Imase and Waxman 1991]

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- Sustainability: ski-rental problem

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Common in operations research and computer science:

- Resource management: scheduling, packing and load balancing problems
- Dynamic data structures: list access problem
- Memory management: paging problem
- Sustainability: ski-rental problem
- Network design: online versions of Steiner tree and facility location problems

Online Load Balancing problem

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Input: machines M, tasks D, sizes $s : D \to \mathbb{R}^+$

Solution: assignment of tasks to machines

Cost: $\max_{i=1}^{M} l(i)$

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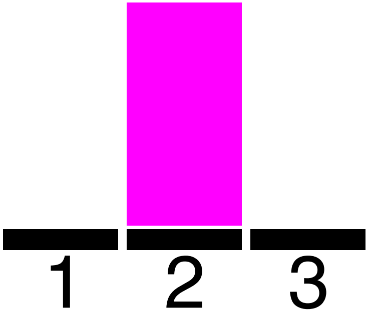

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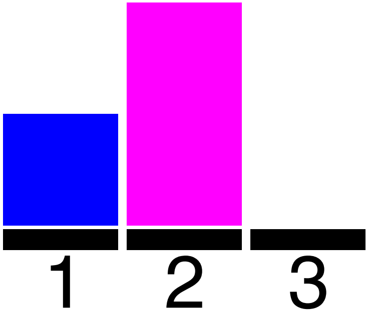

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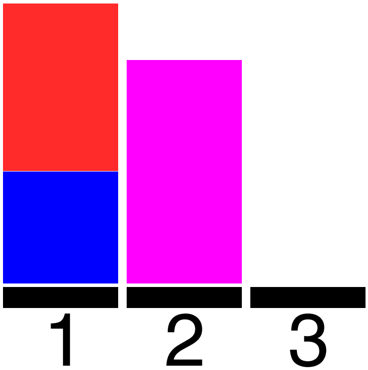

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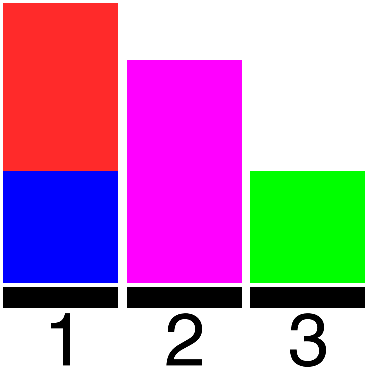

Greedy Online Load Balancing Algorithm

Greedy Online Load Balancing Algorithm

Algorithm 2: OLB Algorithm

Input: M

For each machine $i = 1, \ldots, M$ set its load $l(i)$ to 0;

$i^* \leftarrow 1$;

while *a new task j arrives* **do**

  $a(j) \leftarrow i^*$;

  $l(i^*) \leftarrow l(i^*) + s(j)$;

  choose machine with minimum load as new i^*;

end

return a;

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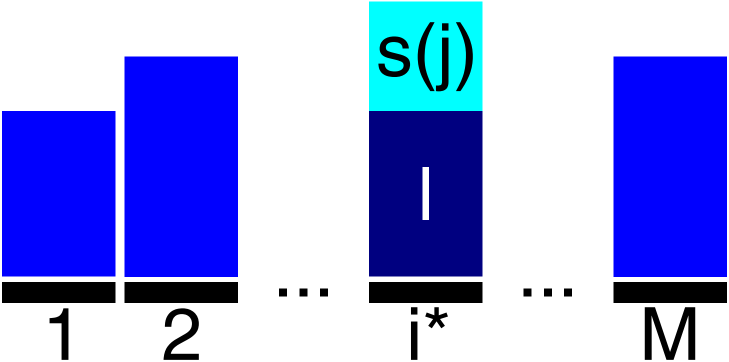

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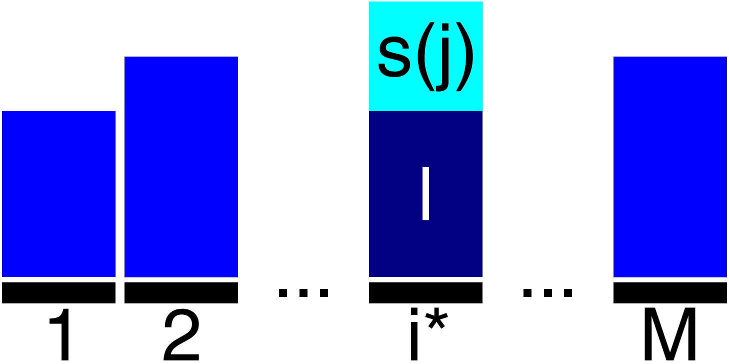

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$$\begin{aligned} \mathrm{ALG} &= l + s(j) \\ &\leq \mathrm{OPT} - \frac{s(j)}{M} + s(j) \\ &\leq \mathrm{OPT} + \left(1 - \frac{1}{M}\right)\mathrm{OPT} \\ &= \left(2 - \frac{1}{M}\right)\mathrm{OPT} \end{aligned}$$

Lower Bound for OLB Algorithm

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List with $M(M-1)$ size 1 tasks followed by one size M task

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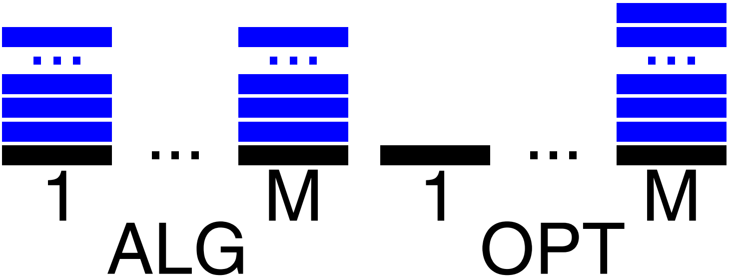

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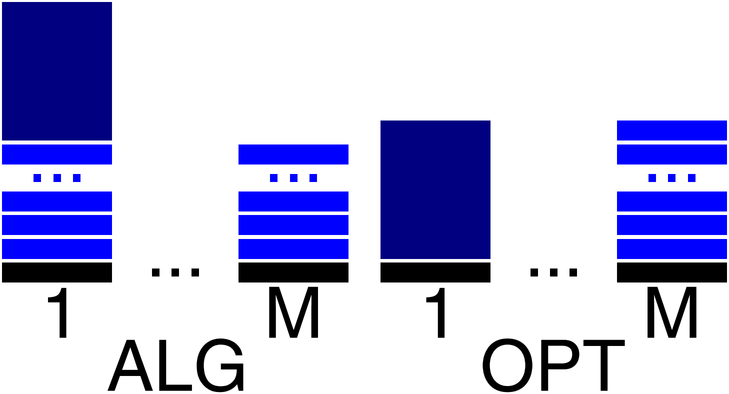

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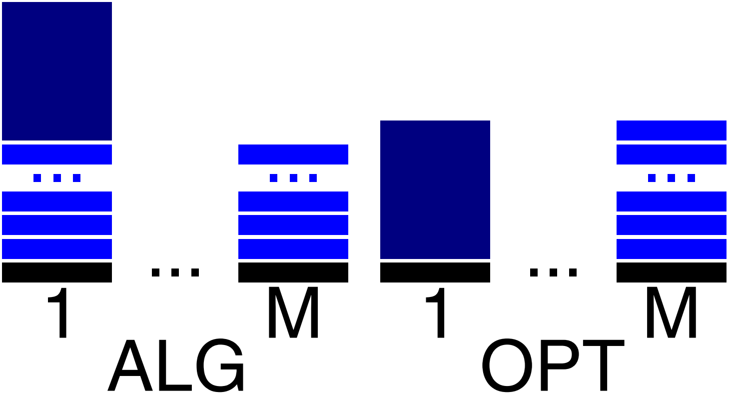

We have $\mathrm{ALG} = 2M - 1$ and $\mathrm{OPT} = M$

Ski Rental Problem

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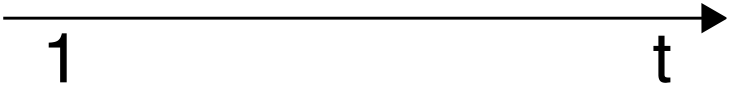
Minimization problem
Input: skis price M, list informing when snow melts
Solution: list informing when we rent or buy skis
Cost: 1 for each renting day plus M if we buy skis



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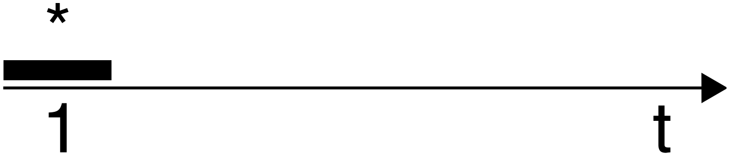

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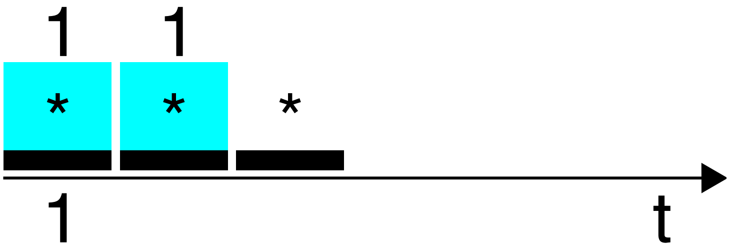

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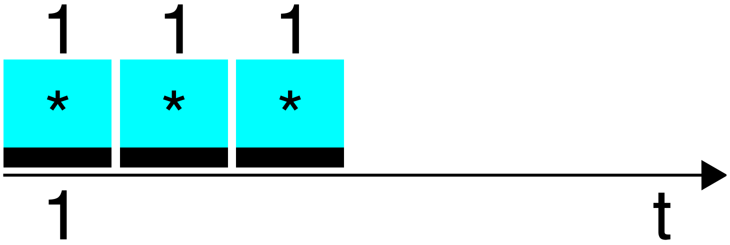

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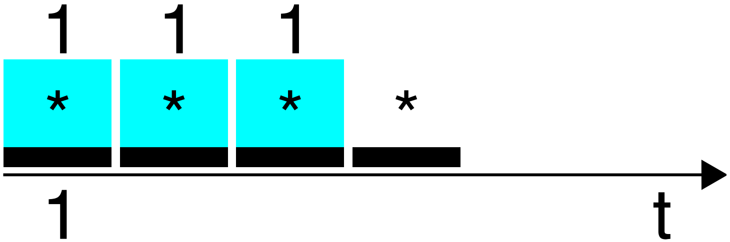

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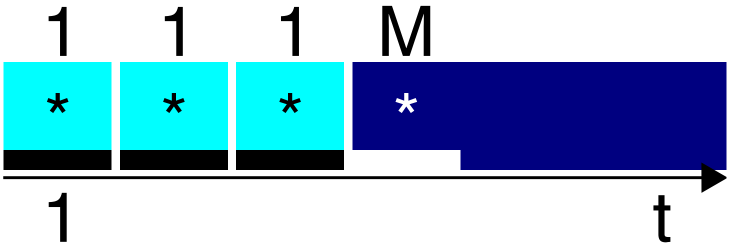

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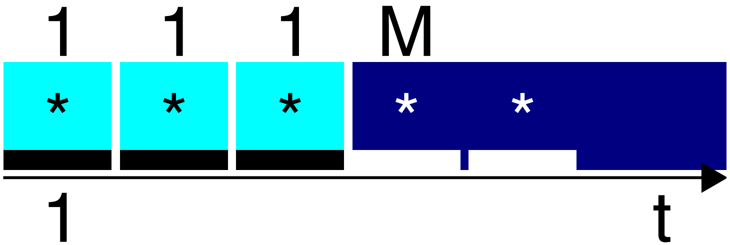

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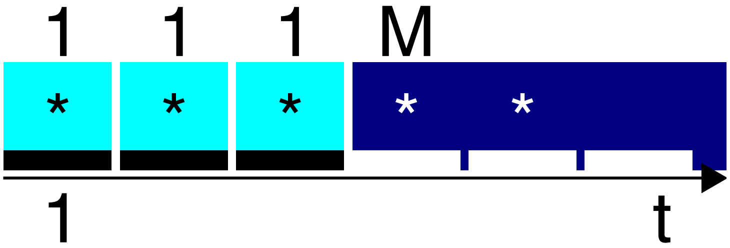

Ski Rental Application and Generalization

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Ski rental algorithms useful to save energy
Help to decide when to turn off parts of a system
Like cores in a processor or computers in a cluster

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Generalized into Parking Permit Problem [Meyerson 2005]
Important to theoretical and practical leasing problems

Ski Rental Algorithm

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Algorithm 3: Intuitive SR Algorithm

Input: M

Set day j and total renting cost r to 0;

while *a new snow day happens* **do**

    if $r + 1 < M$ **then**

        Rent skis at day j and $r \leftarrow r + 1$;

    else

        Buy skis if still don't have them;

    end

    $j \leftarrow j + 1$;

end

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This algorithm is 2-competitive. Why?

Ski Rental LP Formulations

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Linear programming relaxation

$$\begin{aligned} \min \quad & Mx + \sum_{j=1}^{n} y_j & \\ \text{s.t.} \quad & x + y_j \geq 1 & \text{for } j = 1, \ldots, n \\ & x \geq 0, y_j \geq 0 & \text{for } j = 1, \ldots, n \end{aligned}$$

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and its dual

$$\begin{aligned} \max \quad & \textstyle\sum_{j=1}^{n} \alpha_j \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} \alpha_j \leq M \\ & \alpha_j \leq 1 && \text{for } j = 1, \dots, n \\ & \alpha_j \geq 0 && \text{for } j = 1, \dots, n \end{aligned}$$

Primal-Dual Ski Rental Algorithm

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Algorithm 4: Primal-Dual SR Algorithm

Input: M

Set day j' to 0;

while *a new snow day happens* **do**

  increase $\alpha_{j'}$ until one of the following happens:

  (a) $\alpha_{j'} = 1$; /* rent skis setting $y_{j'} = 1$ */

  (b) $M = \alpha_{j'} + \sum_{j=1}^{j'-1} \alpha_j$; /* buy skis setting $x = 1$ */

  $j' \leftarrow j' + 1$;

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Primal-Dual Ski Rental Algorithm

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Is it similar to the previous algorithm?

Primal-Dual SR Algorithm is 2-Competitive

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Cost of any dual solution is at most OPT

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So

$$\begin{aligned} ALG &= Mx + \sum_{j=1}^{n} y_j \\ &\leq \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \alpha_j \\ &\leq 2\mathrm{OPT} \end{aligned}$$

Online Facility Location Problem

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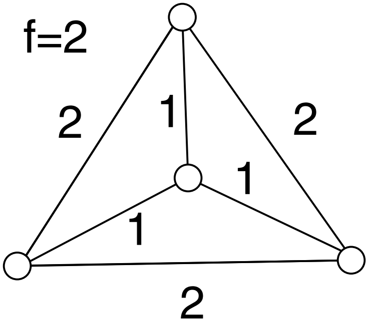

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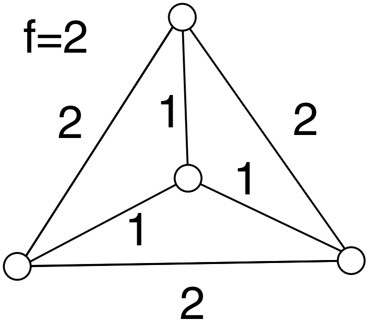

$$\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j))$$

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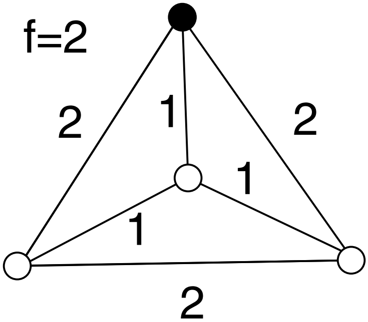

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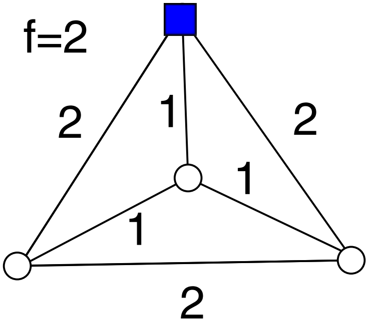

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Total cost $= 2$

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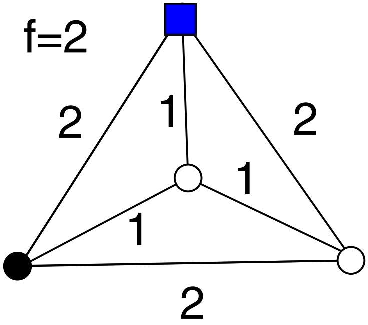

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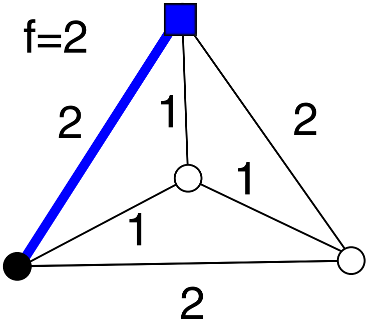

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Total cost $= 2 + 2$

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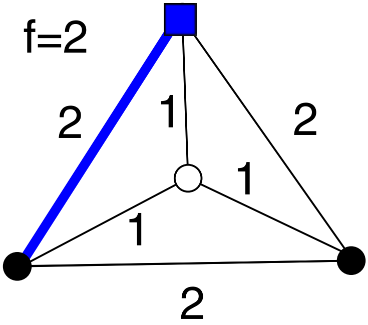

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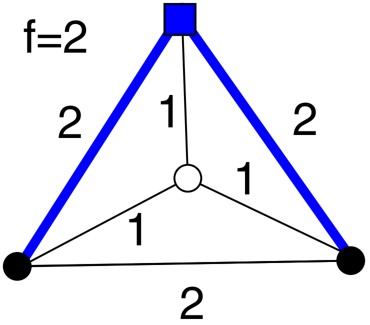
Online Facility Location Problem



$$\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j))$$

Total cost $= 2 + 2 + 2$

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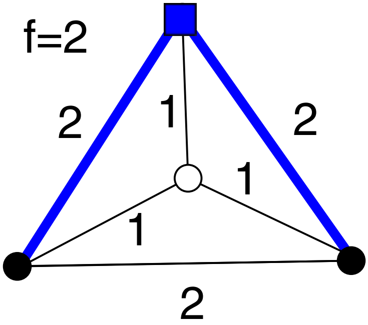

$$\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j))$$

Total cost $= 2 + 2 + 2 = 6.$

Online Facility Location LP Formulation

Online Facility Location LP Formulation

Linear programming relaxation

$$
\begin{aligned}
\min \quad & \sum_{i\in F} f(i)y_i + \sum_{j\in D}\sum_{i\in F} d(j,i)x_{ji} & \\
\text{s.t.} \quad & x_{ji} \le y_i & \text{for } j\in D \text{ and } i\in F\\
& \sum_{i\in F} x_{ji} \ge 1 & \text{for } j\in D\\
& y_i \ge 0, x_{ji} \ge 0 & \text{for } j\in D \text{ and } i\in F
\end{aligned}
$$

Online Facility Location LP Formulation

Linear programming relaxation

$$\begin{aligned} \min \quad & \textstyle\sum_{i \in F} f(i) y_i + \sum_{j \in D} \sum_{i \in F} d(j,i) x_{ji} & \\ \text{s.t.} \quad & x_{ji} \leq y_i & \text{for } j \in D \text{ and } i \in F \\ & \textstyle\sum_{i \in F} x_{ji} \geq 1 & \text{for } j \in D \\ & y_i \geq 0, x_{ji} \geq 0 & \text{for } j \in D \text{ and } i \in F \end{aligned}$$

and its dual

$$\begin{aligned} \max \quad & \textstyle\sum_{j \in D} \alpha_j & \\ \text{s.t.} \quad & \textstyle\sum_{j \in D} (\alpha_j - d(j,i))^+ \leq f(i) & \text{for } i \in F \\ & \alpha_j \geq 0 & \text{for } j \in D \end{aligned}$$

Online Facility Location Algorithm

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Algorithm 5: OFL Algorithm

Input: (G, d, f, F)
$F^a \leftarrow \emptyset$; $D \leftarrow \emptyset$;
while *a new client j' arrives* **do**
  increase $\alpha_{j'}$ until one of the following happens:
  (a) $\alpha_{j'} = d(j', i)$ for some $i \in F^a$; /* connect only */
  (b) $f(i) = (\alpha_{j'} - d(j', i)) + \sum_{j \in D}(d(j, F^a) - d(j, i))^+$ for some $i \in F \setminus F^a$; /* open and connect */
  $F^a \leftarrow F^a \cup \{i\}$; $D \leftarrow D \cup \{j'\}$; $a(j') \leftarrow i$;
end
return (F^a, a);

Online Facility Location Results

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[Nagarajan and Williamson 2013] give a dual-fitting analysis for the algorithm by [Fotakis 2007]

Acknowledgements

Thank you!

Questions?