Online Combinatorial Optimization Problems

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Combinatorial Optimization Problems

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Maximization or minimization problems

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Algorithm receives an input

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Returns a solution with a cost

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As an example, lets take the Steiner Tree Problem

Steiner Tree Problem

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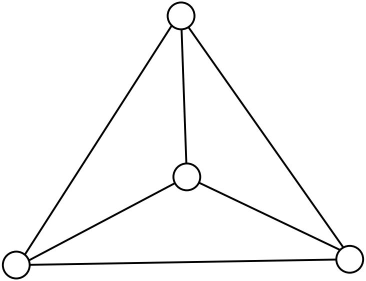
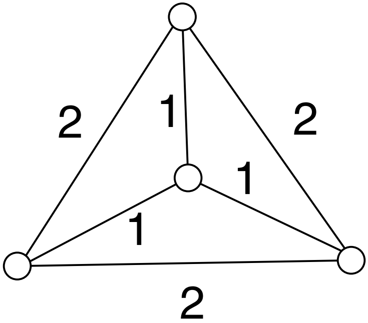
Input: $G = (V, E)$



Steiner Tree Problem

Input: $G = (V, E)$, $d : E \to \mathbb{R}^+$



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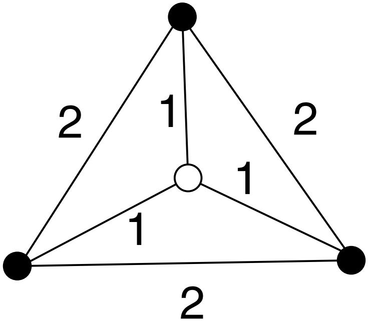
Input: $G = (V, E)$, $d : E \to \mathbb{R}^+$, terminals $D \subseteq V$



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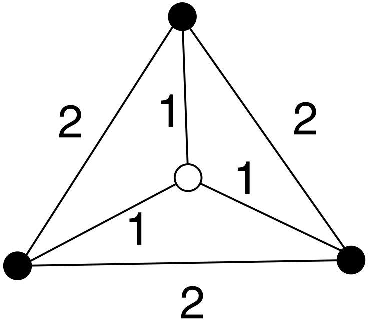

$$\min \sum_{e \in T} d(e)$$

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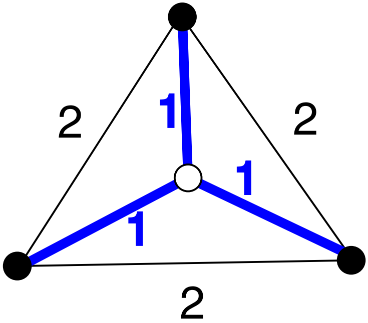

$$\min \sum_{e \in T} d(e)$$

Total cost $= 3$

Online Problems

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Input parts arrive one at a time

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Each part is served before next one arrives

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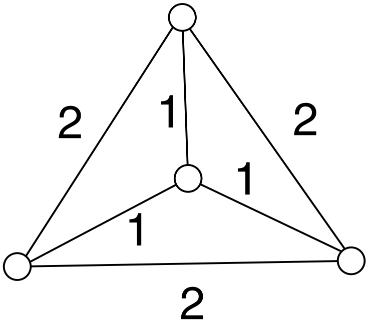
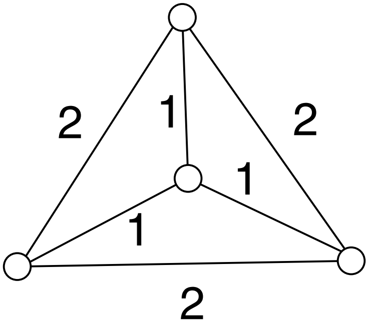


Online Steiner Tree Problem

Terminal nodes arrive one at a time



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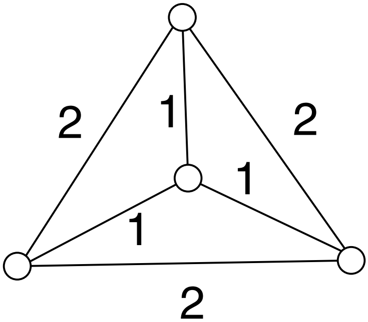

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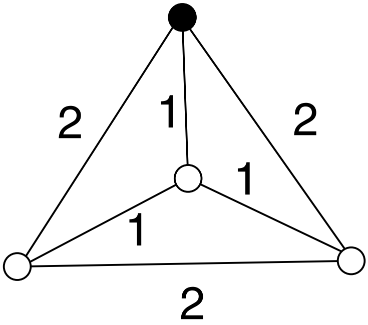

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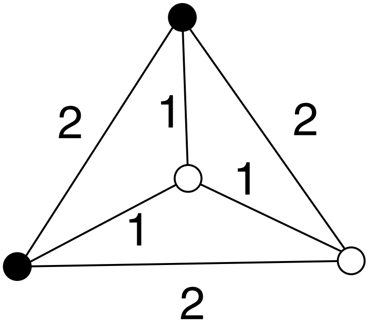

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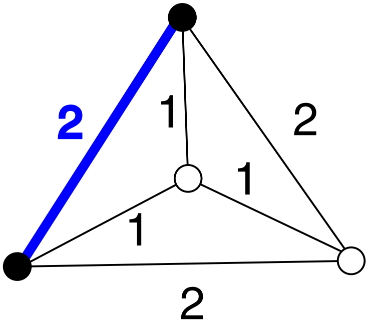

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Total cost $= 2$

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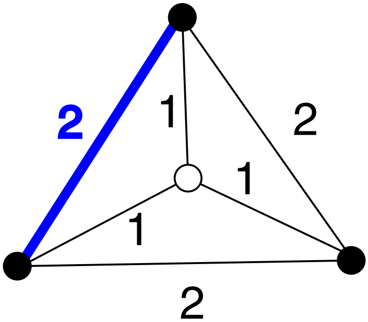

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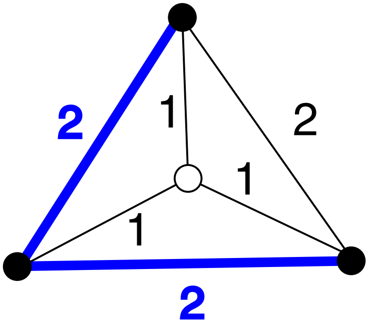

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Total cost $= 2 + 2$

Online Steiner Tree Problem

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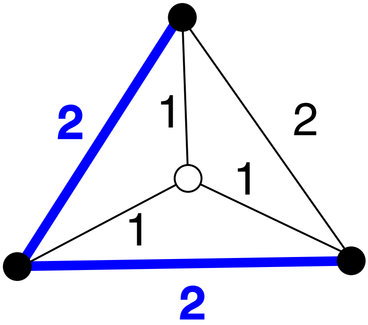
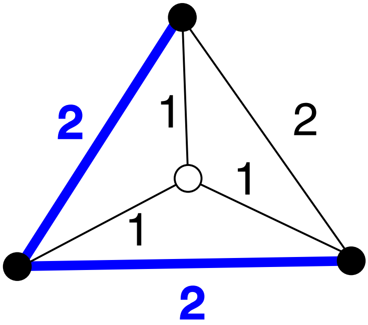


$$\min \sum_{e \in T} d(e)$$

Total cost $= 2 + 2 = 4$

Online Steiner Tree Problem

Terminal nodes arrive one at a time
No edge used can be removed in the future



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Competitive Analysis

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Worst case analysis technique

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For online algorithm ALG

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Using offline optimal solution OPT

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ALG is c-competitive if

$$\mathrm{ALG}(I) \leq c\,\mathrm{OPT}(I)$$

for every input I

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Worst case analysis technique

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As an example, lets take a greedy online algorithm for the Online Steiner Tree problem

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Input: (G, d)

$T \leftarrow (\emptyset, \emptyset)$;

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A $\Omega(\log n)$ lower bound is known [Imase and Waxman 1991]

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- Resource management: scheduling, packing and load balancing problems
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- Memory management: paging problem
- Sustainability: ski-rental problem
- Network design: Steiner tree and facility location problems

Online Load Balancing problem

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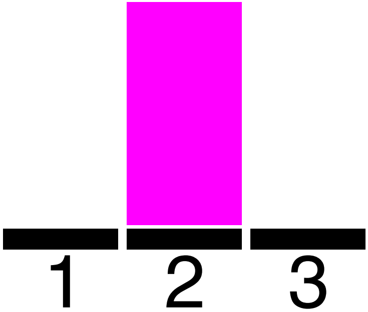

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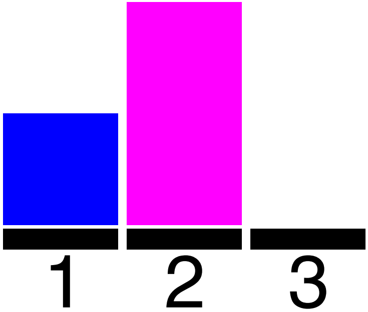

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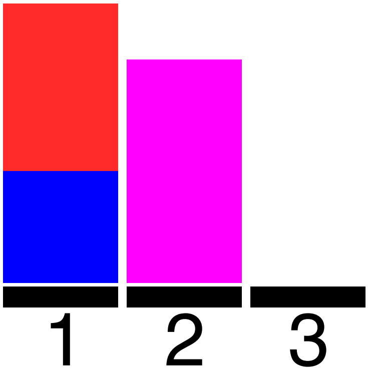

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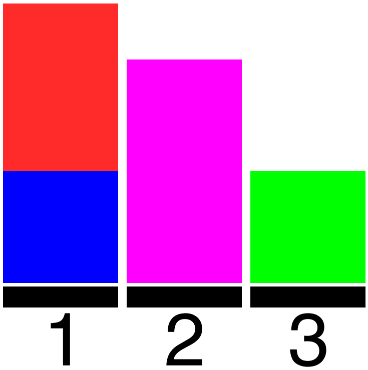

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For each machine $i = 1, \ldots, M$ set its load $l(i)$ to 0;

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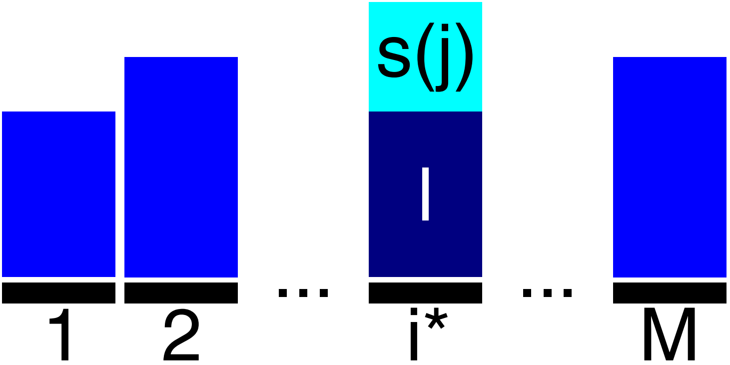
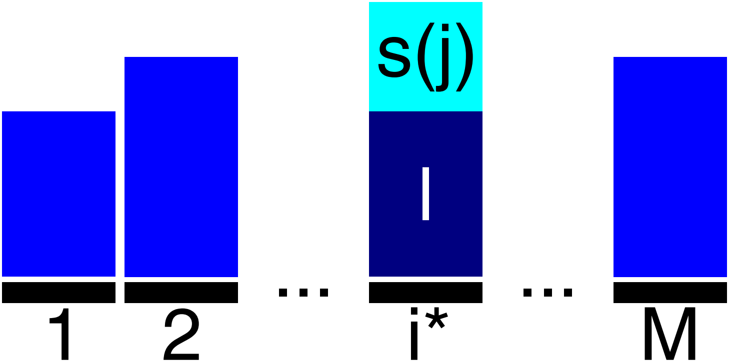


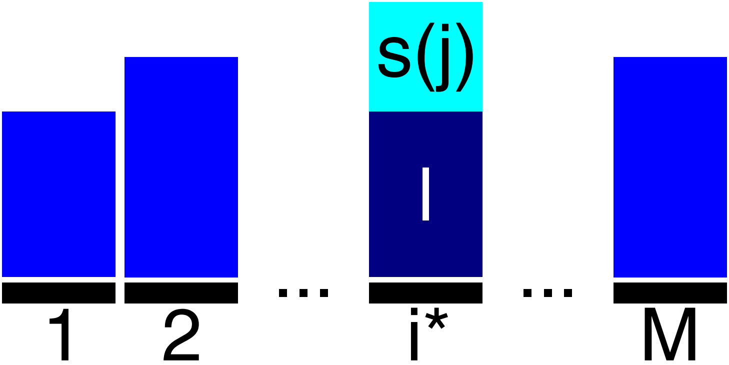
OLB Algorithm is $\left(2 - \frac{1}{M}\right)$-competitive

Let i^* be the machine with maximum load



OLB Algorithm is $\left(2 - \frac{1}{M}\right)$-competitive

Let i^* be the machine with maximum load, j be the last task assigned to i^*



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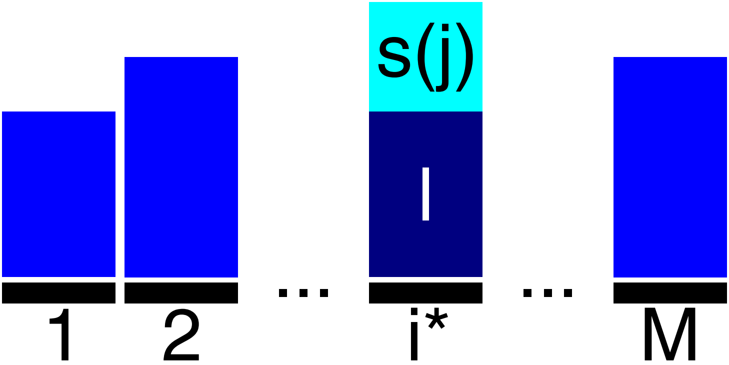
Let i^* be the machine with maximum load, j be the last task assigned to i^*, and $l(i^*) = l + s(j)$



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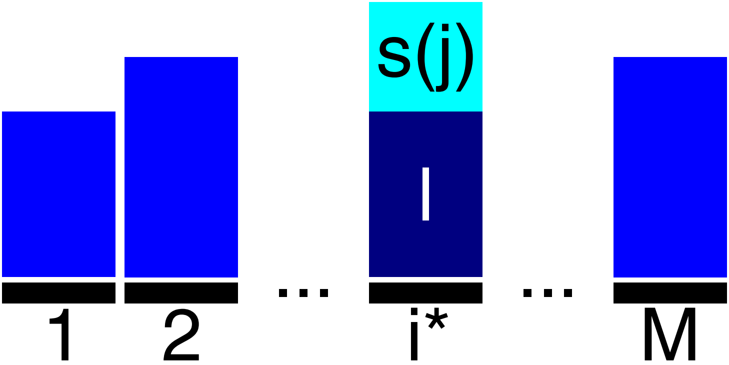

We have $\mathrm{OPT} \geq s(j)$ and $\mathrm{OPT} \geq l + \frac{s(j)}{M}$

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$$\mathrm{ALG} = l + s(j)$$

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Since $\mathrm{OPT} \geq s(j)$ and $\mathrm{OPT} \geq l + \frac{s(j)}{M}$, we have

$$\begin{aligned} \mathrm{ALG} &= l + s(j) \\ &\leq \mathrm{OPT} - \frac{s(j)}{M} + s(j) \end{aligned}$$

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Lower Bound for OLB Algorithm

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List with $M(M-1)$ size 1 tasks

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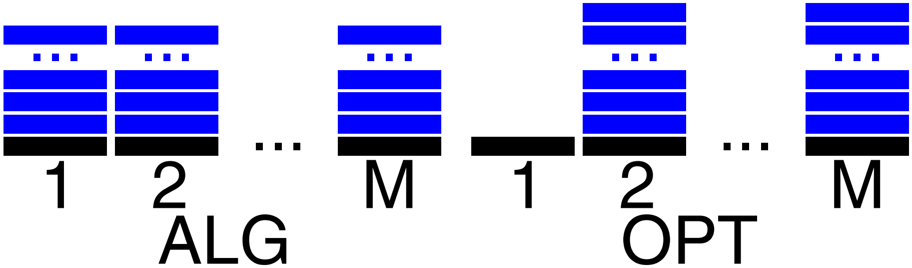
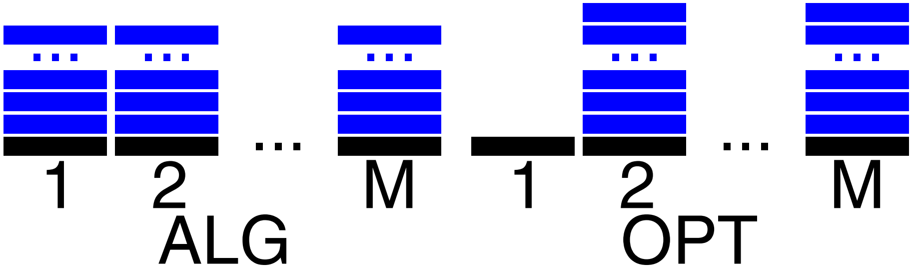


Lower Bound for OLB Algorithm

List with $M(M-1)$ size 1 tasks followed by one size M task



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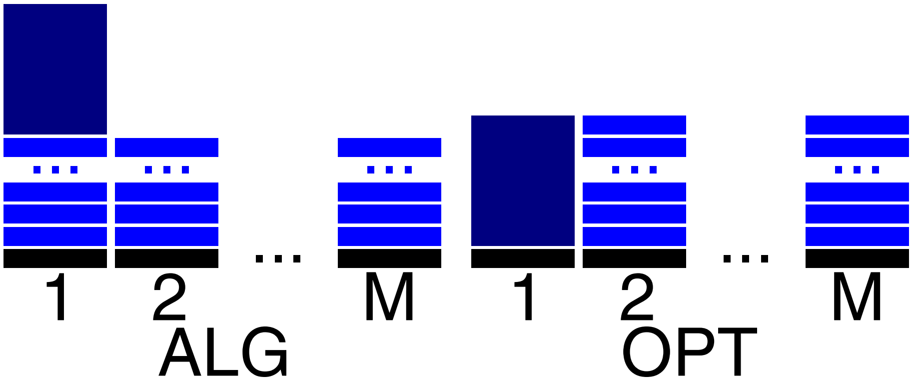

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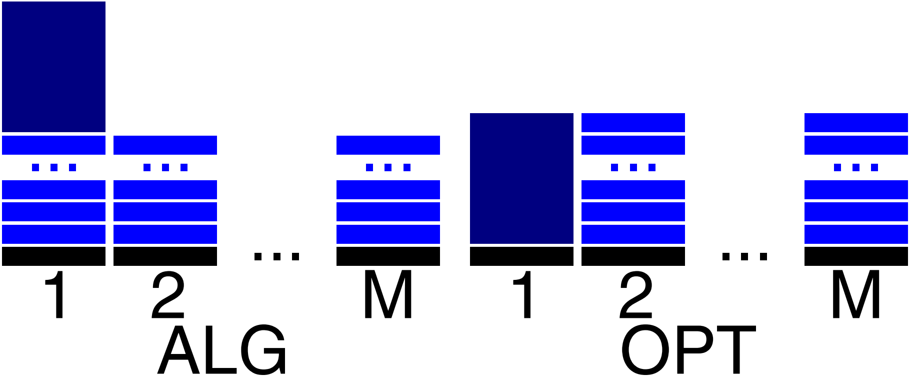

We have $\mathrm{ALG} = 2M - 1$ and $\mathrm{OPT} = M$

Ski Rental Problem

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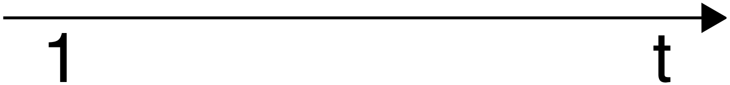
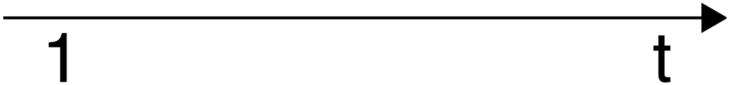
Input: time horizon



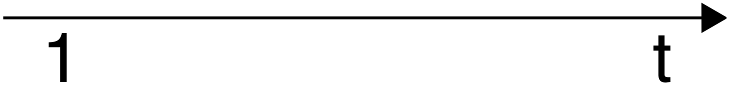
Ski Rental Problem

Input: time horizon, skis buying price M (renting cost is 1 per day)



Ski Rental Problem

Input: time horizon, skis buying price M (renting cost is 1 per day), list informing when snow melts



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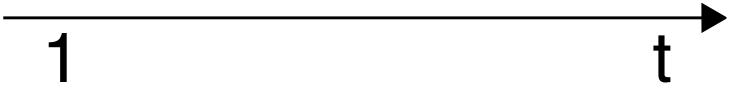

minimize sum of renting days

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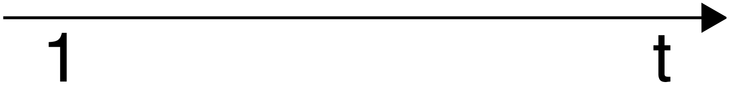

minimize sum of renting days plus M (if we decide to buy skis)

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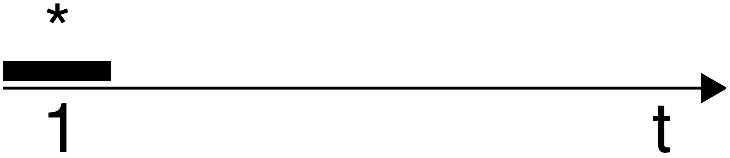
Input: time horizon, skis buying price M (renting cost is 1 per day), list informing when snow melts



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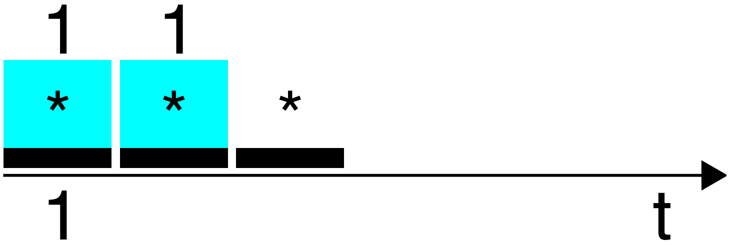

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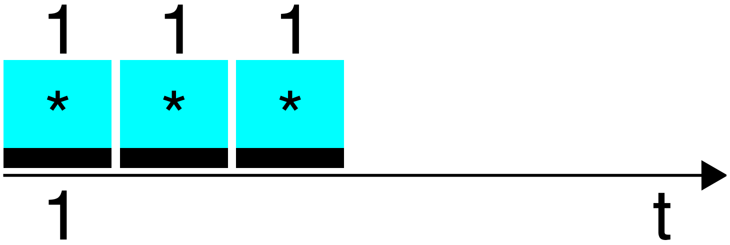

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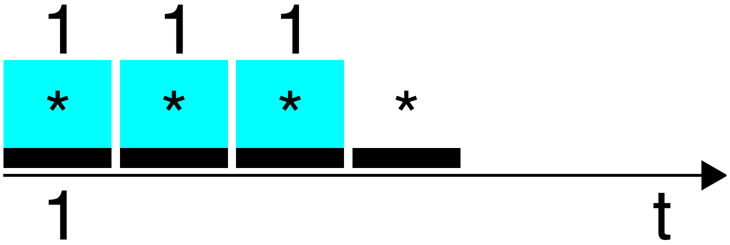
Ski Rental Problem

Input: time horizon, skis buying price M (renting cost is 1 per day), list informing when snow melts



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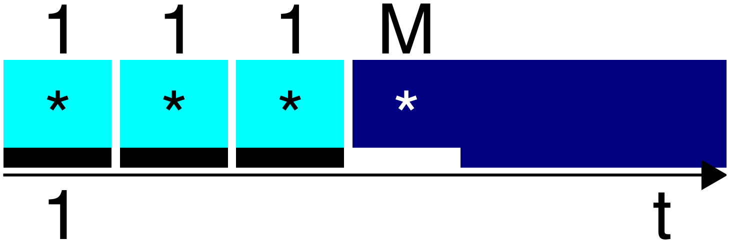

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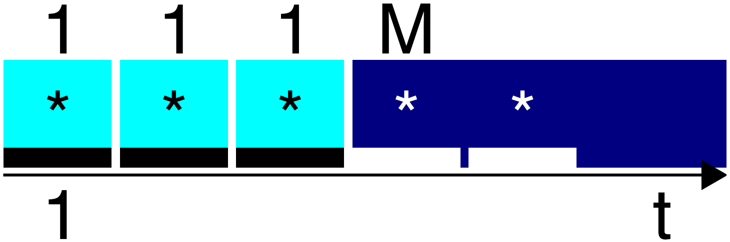

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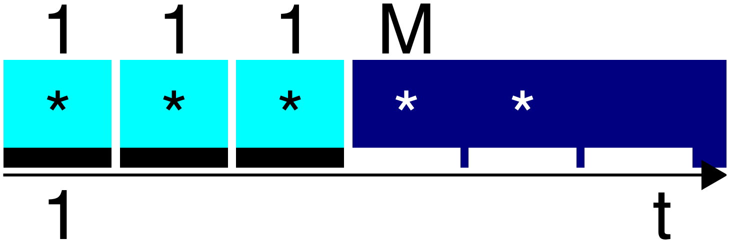

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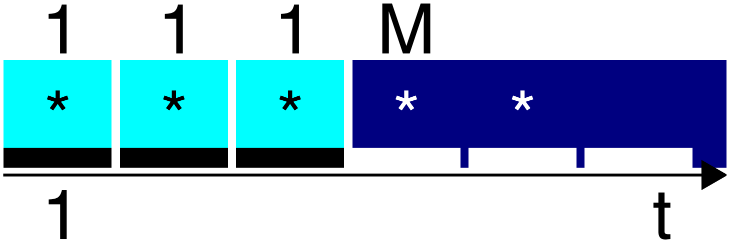

minimize sum of renting days plus M (if we decide to buy skis)

Does a greedy algorithm solve this problem?

Ski Rental Application and Generalization

Ski Rental Application and Generalization

Ski rental algorithms are useful to save energy

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Help to decide when to turn off parts of a system

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Like cores in a processor or computers in a cluster

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Generalized into Parking Permit Problem [Meyerson 2005]

Ski Rental Application and Generalization

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Important both to theoretical and practical leasing problems

Ski Rental Algorithm

Ski Rental Algorithm

Algorithm 3: Intuitive SR Algorithm

Input: M

Set day j and total renting cost r to 0;

Ski Rental Algorithm

Algorithm 3: Intuitive SR Algorithm

Input: M

Set day j and total renting cost r to 0;

while *a new snow day happens* **do**

Ski Rental Algorithm

Algorithm 3: Intuitive SR Algorithm

Input: M

Set day j and total renting cost r to 0;

while *a new snow day happens* **do**

  if $r + 1 < M$ **then**

    Rent skis at day j and $r \leftarrow r + 1$;

Ski Rental Algorithm

Algorithm 3: Intuitive SR Algorithm

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Set day j and total renting cost r to 0;

while *a new snow day happens* **do**

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  else

    Buy skis if still don't have them;

Ski Rental Algorithm

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Ski Rental Algorithm

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This algorithm is 2-competitive. Why?

Ski Rental LP Formulations

Ski Rental LP Formulations

Linear programming relaxation

Ski Rental LP Formulations

Linear programming relaxation

$$\min \quad Mx + \sum_{j=1}^{n} y_j$$

Ski Rental LP Formulations

Linear programming relaxation

$$\begin{array}{lll} \text{min} & Mx + \sum_{j=1}^{n} y_j & \\ \text{s.t.} & x + y_j \geq 1 & \text{for } j = 1, \ldots, n \end{array}$$

Ski Rental LP Formulations

Linear programming relaxation

$$
\begin{array}{lll}
\min & Mx + \sum_{j=1}^{n} y_j & \\
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\end{array}
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$$\text{max} \quad \sum_{j=1}^{n} \alpha_j$$

Ski Rental LP Formulations

Linear programming relaxation

$$\begin{aligned} \min \quad & Mx + \textstyle\sum_{j=1}^{n} y_j \\ \text{s.t.} \quad & x + y_j \geq 1 && \text{for } j = 1, \ldots, n \\ & x \geq 0, y_j \geq 0 && \text{for } j = 1, \ldots, n \end{aligned}$$

and its dual

$$\begin{aligned} \max \quad & \textstyle\sum_{j=1}^{n} \alpha_j \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} \alpha_j \leq M \end{aligned}$$

Ski Rental LP Formulations

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Primal-Dual Ski Rental Algorithm

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Algorithm 4: Primal-Dual SR Algorithm

Input: M

Set day j' to 0;

Primal-Dual Ski Rental Algorithm

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increase $\alpha_{j'}$ until one of the following happens:

Primal-Dual Ski Rental Algorithm

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increase $\alpha_{j'}$ until one of the following happens:

(a) $\alpha_{j'} = 1$; /* rent skis setting $y_{j'} = 1$ */

Primal-Dual Ski Rental Algorithm

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  increase $\alpha_{j'}$ until one of the following happens:

  (a) $\alpha_{j'} = 1$; /* rent skis setting $y_{j'} = 1$ */

  (b) $M = \alpha_{j'} + \sum_{j=1}^{j'-1} \alpha_j$; /* buy skis setting $x = 1$ */

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Primal-Dual Ski Rental Algorithm

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Is it similar to the previous algorithm?

Primal-Dual SR Algorithm is 2-Competitive

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Cost of any dual solution is at most OPT

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So

$$ALG = Mx + \sum_{j=1}^{n} y_j$$

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ALG &= Mx + \sum_{j=1}^{n} y_j \\
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Primal-Dual SR Algorithm is 2-Competitive

Cost of any dual solution is at most OPT

So

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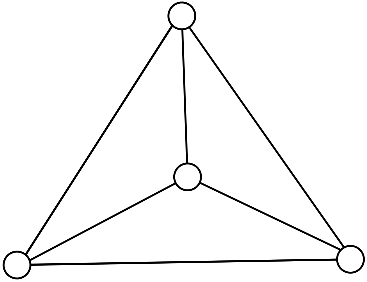
\begin{aligned}
ALG &= Mx + \sum_{j=1}^{n} y_j \\
&\leq \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \alpha_j \\
&\leq 2\mathrm{OPT}
\end{aligned}
$$

Online Facility Location Problem

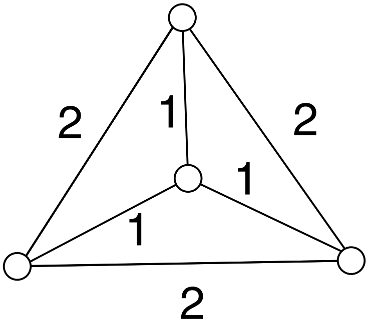
Online Facility Location Problem

Input: $G = (V, E)$



Online Facility Location Problem

Input: $G = (V, E)$, $d : E \to \mathbb{R}^+$



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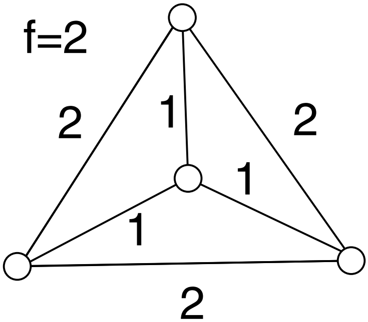

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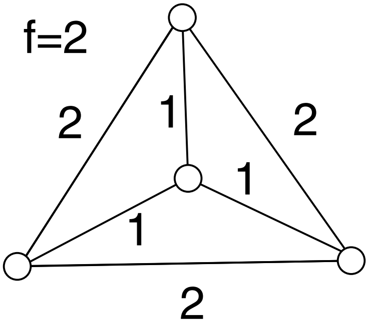
Input: $G = (V, E)$, $d : E \to \mathbb{R}^+$, $f : V \to \mathbb{R}^+$, clients $D \subseteq V$



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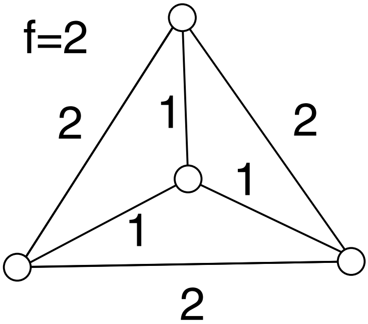

$$\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j))$$

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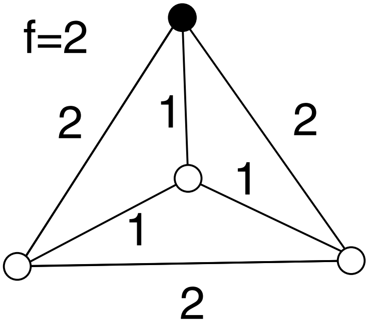

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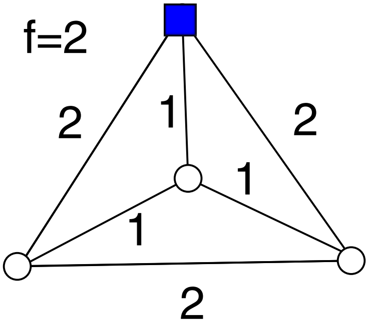

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Total cost $= 2$

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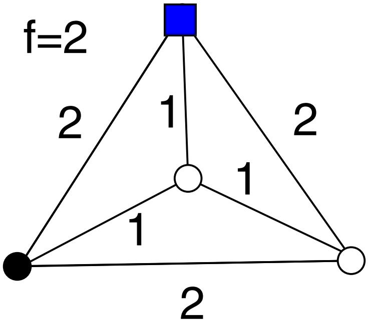

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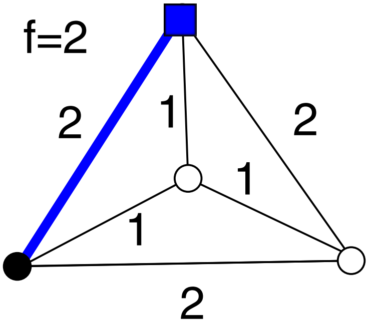

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Total cost $= 2 + 2$

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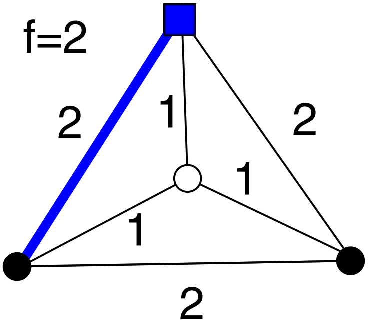

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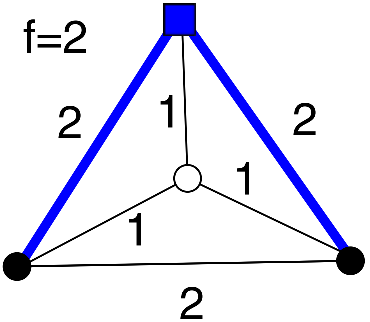

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Total cost $= 2 + 2 + 2$

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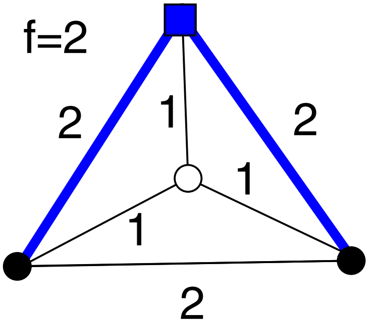

$$\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j))$$

Total cost $= 2 + 2 + 2 = 6.$

Online Facility Location LP Formulation

Online Facility Location LP Formulation

Linear programming relaxation

Online Facility Location LP Formulation

Linear programming relaxation

$$\min \quad \sum_{i \in F} f(i) y_i + \sum_{j \in D} \sum_{i \in F} d(j,i) x_{ji}$$

Online Facility Location LP Formulation

Linear programming relaxation

$$\begin{array}{ll} \text{min} & \sum_{i \in F} f(i) y_i + \sum_{j \in D} \sum_{i \in F} d(j,i) x_{ji} \\ \text{s.t.} & x_{ji} \leq y_i \qquad\qquad \text{for } j \in D \text{ and } i \in F \end{array}$$

Online Facility Location LP Formulation

Linear programming relaxation

$$
\begin{array}{rll}
\min & \sum_{i \in F} f(i) y_i + \sum_{j \in D} \sum_{i \in F} d(j,i) x_{ji} & \\
\text{s.t.} & x_{ji} \leq y_i & \text{for } j \in D \text{ and } i \in F \\
 & \sum_{i \in F} x_{ji} \geq 1 & \text{for } j \in D
\end{array}
$$

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Linear programming relaxation

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\min & \sum_{i \in F} f(i) y_i + \sum_{j \in D} \sum_{i \in F} d(j,i) x_{ji} & \\
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and its dual

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Linear programming relaxation

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$$
\max \quad \sum_{j \in D} \alpha_j
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Online Facility Location LP Formulation

Linear programming relaxation

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and its dual

$$\begin{array}{rll} \max & \sum_{j \in D} \alpha_j & \\ \text{s.t.} & \sum_{j \in D} (\alpha_j - d(j,i))^+ \leq f(i) & \text{for } i \in F \end{array}$$

Online Facility Location LP Formulation

Linear programming relaxation

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\begin{array}{lll}
\min & \sum_{i \in F} f(i) y_i + \sum_{j \in D} \sum_{i \in F} d(j,i) x_{ji} & \\
\text{s.t.} & x_{ji} \leq y_i & \text{for } j \in D \text{ and } i \in F \\
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& \alpha_j \geq 0 & \text{for } j \in D
\end{array}
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Online Facility Location Algorithm

Online Facility Location Algorithm

Algorithm 5: OFL Algorithm

Input: (G, d, f, F)
$F^a \leftarrow \emptyset$; $D \leftarrow \emptyset$;

Online Facility Location Algorithm

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Online Facility Location Algorithm

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Online Facility Location Algorithm

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Online Facility Location Algorithm

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  (a) $\alpha_{j'} = d(j', i)$ for some $i \in F^a$; /* connect only */
  (b) $f(i) = (\alpha_{j'} - d(j', i)) + \sum_{j \in D}(d(j, F^a) - d(j, i))^+$ for some $i \in F \setminus F^a$; /* open and connect */

Online Facility Location Algorithm

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  (b) $f(i) = (\alpha_{j'} - d(j', i)) + \sum_{j \in D}(d(j, F^a) - d(j, i))^+$ for some $i \in F \setminus F^a$; /* open and connect */
  $F^a \leftarrow F^a \cup \{i\}$; $D \leftarrow D \cup \{j'\}$; $a(j') \leftarrow i$;

Online Facility Location Algorithm

Algorithm 5: OFL Algorithm

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while *a new client j' arrives* **do**
    increase $\alpha_{j'}$ until one of the following happens:
    (a) $\alpha_{j'} = d(j', i)$ for some $i \in F^a$; /* connect only */
    (b) $f(i) = (\alpha_{j'} - d(j', i)) + \sum_{j \in D}(d(j, F^a) - d(j, i))^+$ for some $i \in F \setminus F^a$; /* open and connect */
    $F^a \leftarrow F^a \cup \{i\}$; $D \leftarrow D \cup \{j'\}$; $a(j') \leftarrow i$;
return (F^a, a);

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[Nagarajan and Williamson 2013] give a dual-fitting analysis for the algorithm by [Fotakis 2007]

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Questions?