Laboratório de Otimização e Combinatória

Instituto de Computação

UNICAMP

Online Combinatorial Optimization Problems

DTC/LOCo

Mário César San Felice

Postdoc at IME-USP

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As an example, lets take the Steiner Tree Problem.

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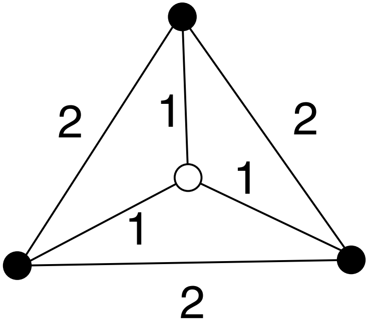
Minimization problem whose input is a graph with costs on the edges, a set of terminal nodes and a set of Steiner nodes. A feasible solution is a tree that connects all terminal nodes, and its cost is the sum of the edge costs in the tree.



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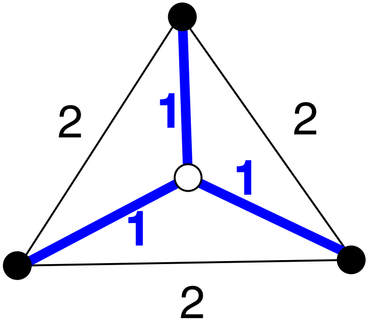

Online Problems

Problems in which the parts of the input arrive one at a time and each part need to be served before the next one arrives. Also, no decision made to serve a part may be changed in the future.

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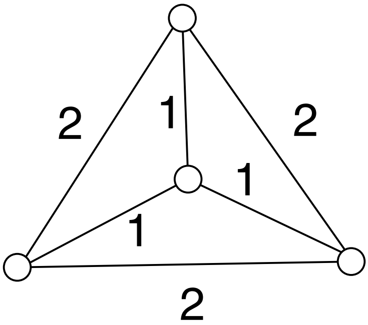

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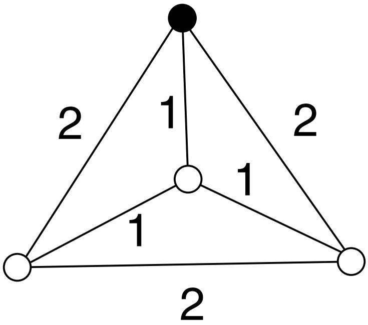

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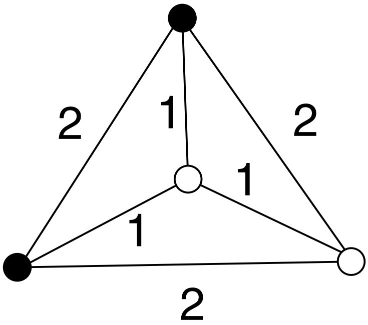

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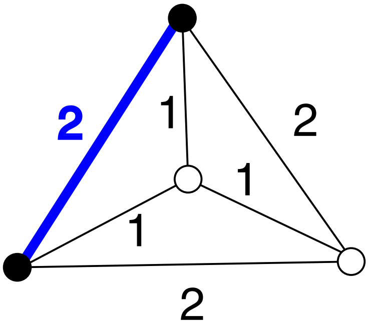

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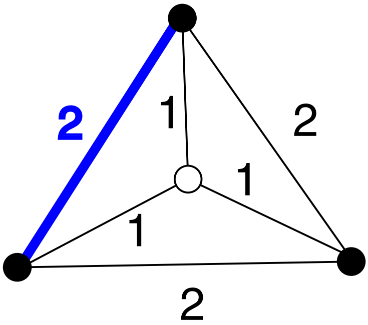

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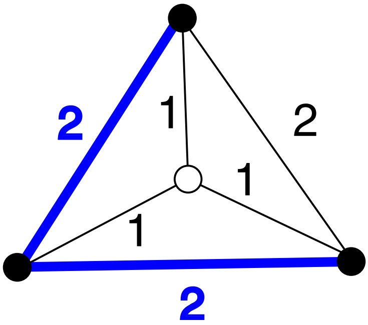

Competitive Analysis

Worst case technique used to analyse online algorithms.

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As an example, lets take a greedy online algorithm for the Online Steiner Tree problem.

Greedy Online Steiner Tree Algorithm

Algorithm 1: OST Algorithm.

Input: (G, d)
$T \leftarrow (\emptyset, \emptyset)$;
while *a new terminal j arrives* **do**
    $T \leftarrow T \cup \{\mathrm{path}(j, V(T))\}$;
end
return T;

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The OST Algorithm is $\mathrm{O}(\log n)$-competitive. Also, it is known a $\Omega(\log n)$ lower bound to the competitive ratio of any algorithm for this problem [Imase and Waxman 1991].

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- Dynamic data structures: list access problem.
- Memory management: paging problem.
- Sustainability: ski-rental and parking permit problems.
- Network design: online versions of Steiner tree and facility location problems.

Online Load Balancing problem

Minimization problem whose input is a number of machines and a list of tasks with sizes. A feasible solution is an assignment of each task to a machine, and its cost is the maximum load between the machines.

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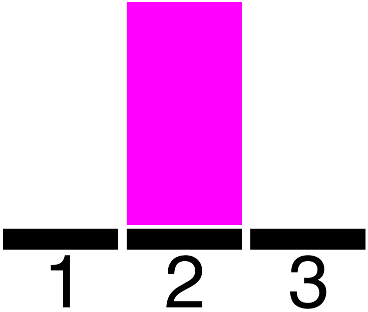

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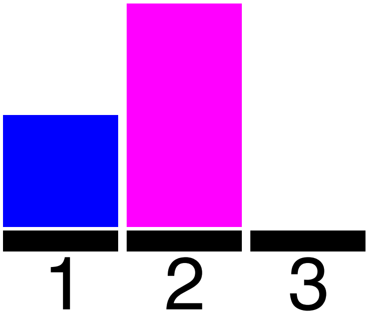

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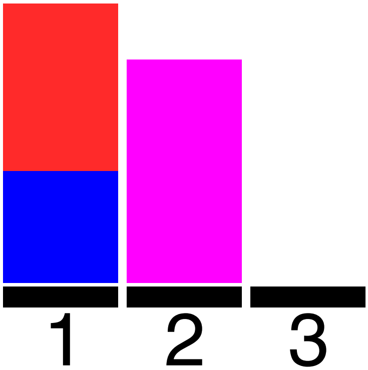

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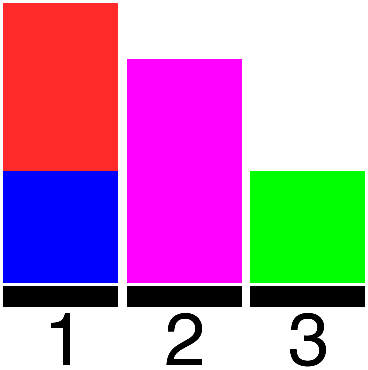

Greed Online Load Balancing Algorithm

Algorithm 2: OLB Algorithm.

Input: M

For each machine $i = 1, \ldots, M$ set its load $l(i)$ to 0;

$i^* \leftarrow 1$;

while *a new task j arrives* **do**

  $a(j) \leftarrow i^*$;

  $l(i^*) \leftarrow l(i^*) + s(j)$;

  choose machine with minimum load as new i^*;

end

return a;

OLB Algorithm is $\left(2 - \frac{1}{M}\right)$-competitive

Let i^* be the machine with the maximum load, j be the last task assigned to i^*, and $l(i^*) = l + s(j)$.

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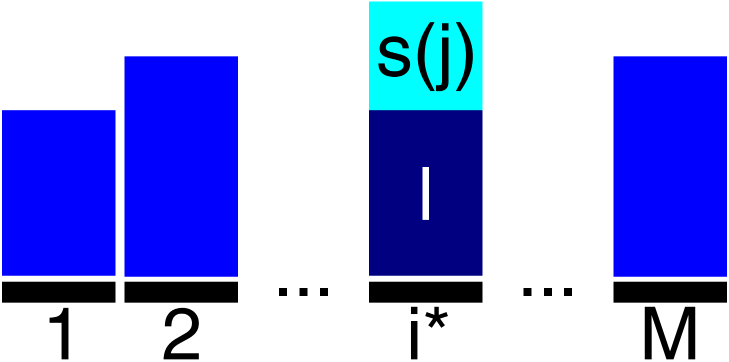

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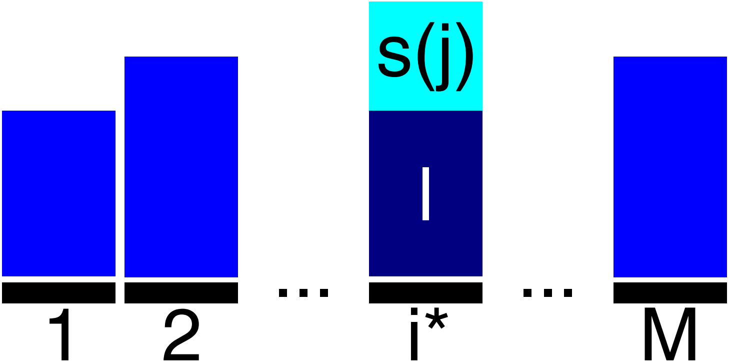

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$$\begin{aligned}
\mathrm{ALG} &= l + s(j) \\
&\leq \mathrm{OPT} - \frac{s(j)}{M} + s(j) \\
&\leq \mathrm{OPT} + \left(1 - \frac{1}{M}\right)\mathrm{OPT} \\
&= \left(2 - \frac{1}{M}\right)\mathrm{OPT}\ .
\end{aligned}$$

Lower Bound for OLB Algorithm

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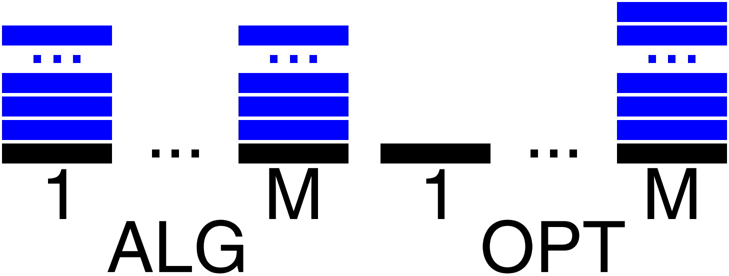

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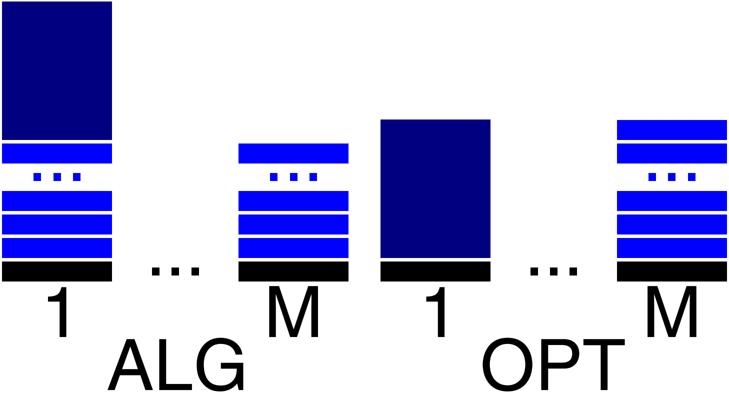

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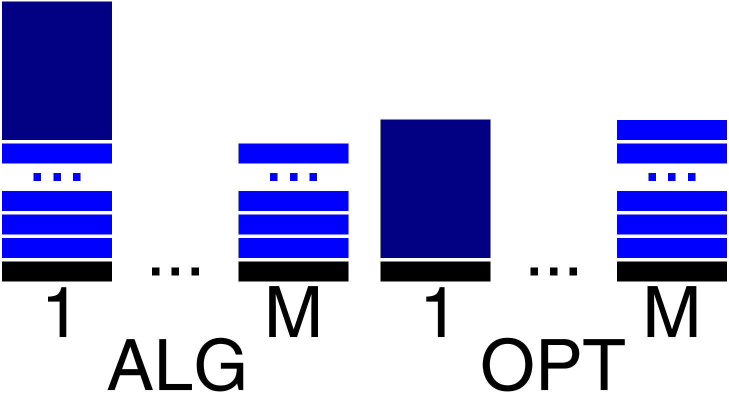
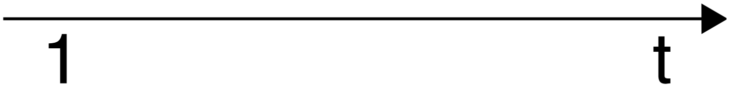


We have $\mathrm{ALG} = 2M - 1$ and $\mathrm{OPT} = M$.

Ski Rental Problem

Minimization problem whose input is the price M to buy skies and a list informing when the snow melt. A feasible solution is a list informing when we rent or buy skies, and its cost is 1 for each renting day plus M if you buy.



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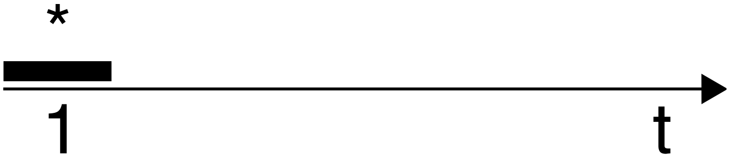

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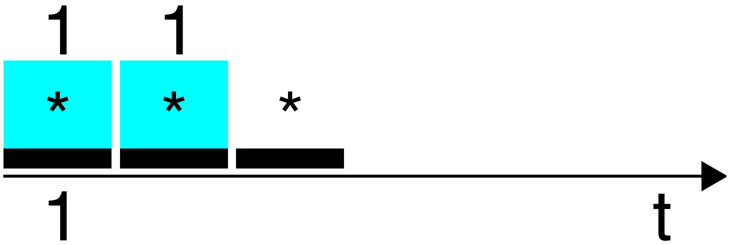

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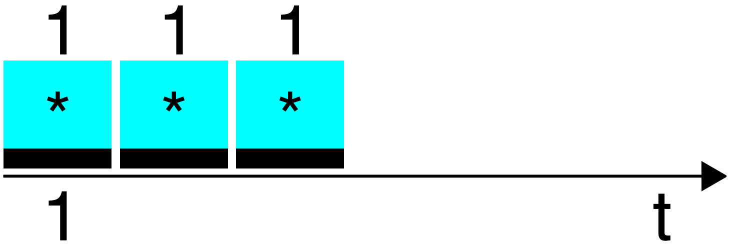

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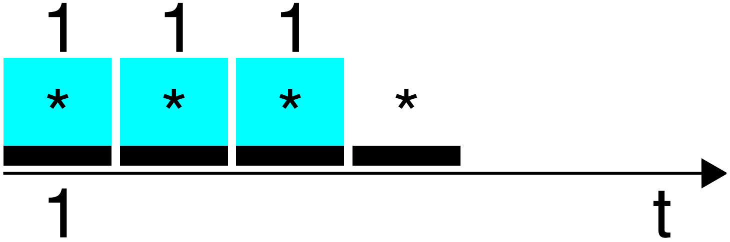

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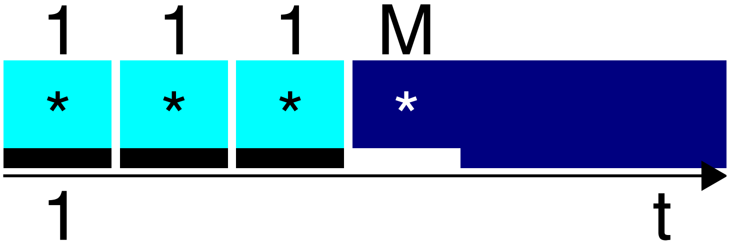

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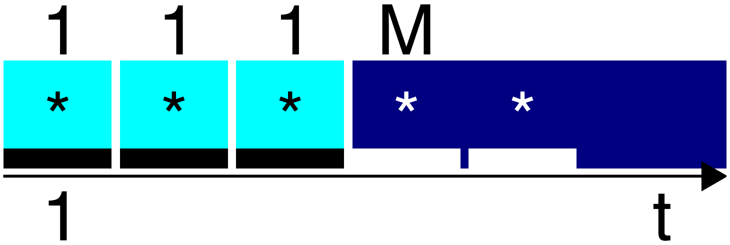

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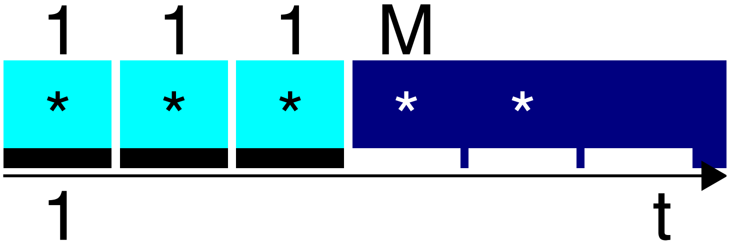

Ski Rental Application and Generalization

Ski rental algorithms may be used to save energy by deciding when to turn off parts of a system, like cores in a processor or computers in a cluster.

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Also, it may be generalized to the Parking Permit Problem [Meyerson 2005].

Ski Rental Algorithm

Algorithm 3: Intuitive SR Algorithm.

Input: M

Set day j and total renting cost r to 0;

while *a new snow day happens* **do**

  if $r + 1 < M$ **then**

    Rent skies at day j and $r \leftarrow r + 1$;

  else

    Buy skies if still don't have them;

  end

  $j \leftarrow j + 1$;

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This algorithm is 2-competitive. Why?

Ski Rental LP Formulations

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Linear programming relaxation

$$
\begin{aligned}
\min \quad & Mx + \textstyle\sum_{j=1}^{n} y_j \\
\text{s.t.} \quad & x + y_j \geq 1 \qquad \text{for } j = 1, \ldots, n, \\
& x \geq 0, y_j \geq 0 \quad \text{for } j = 1, \ldots, n,
\end{aligned}
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and its dual

$$\begin{aligned} \max \quad & \textstyle\sum_{j=1}^{n} \alpha_j & \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} \alpha_j \leq M & \text{for } j = 1, \ldots, n, \\ & \alpha_j \leq 1 & \text{for } j = 1, \ldots, n. \\ & \alpha_j \geq 0 & \text{for } j = 1, \ldots, n. \end{aligned}$$

Primal-Dual Ski Rental Algorithm

Algorithm 4: Primal-Dual SR Algorithm.

Input: M

Set day j' to 0;

while *a new snow day happens* **do**

increase $\alpha_{j'}$ until one of the following happens:

(a) $\alpha_{j'} = 1$; /* rent skies setting $y_j = 1$ */

(b) $M = \alpha_{j'} + \sum_{j=1}^{j'-1} \alpha_j$; /* buy skies setting $x = 1$ */

$j' \leftarrow j' + 1$;

end

Does it remember the previous algorithm?

Primal-Dual SR Algorithm is 2-Competitive

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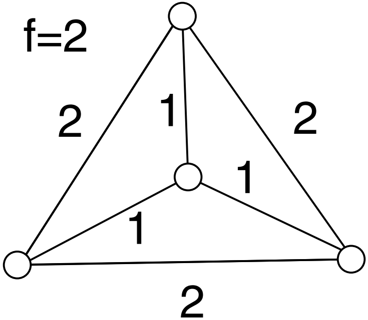
$$\begin{aligned} ALG &= Mx + \sum_{j=1}^{n} y_j \\ &\leq \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \alpha_j \\ &\leq 2\mathrm{OPT} \ . \end{aligned}$$

Online Facility Location Problem



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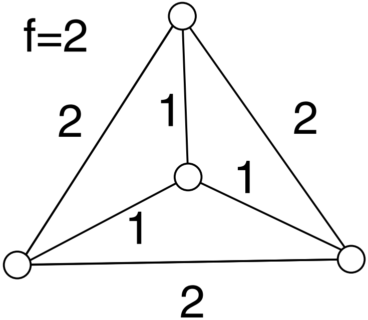

$$\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j))$$

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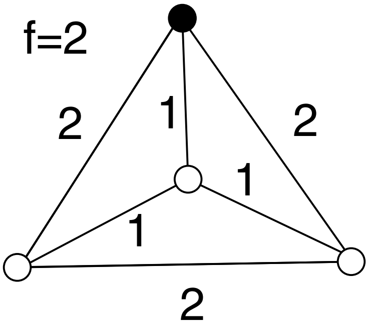

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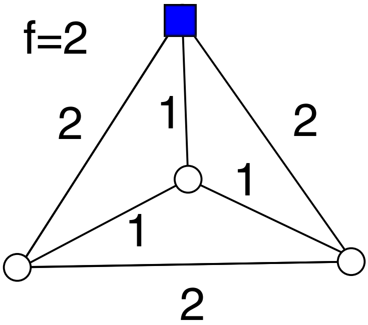

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Total cost $= 2$

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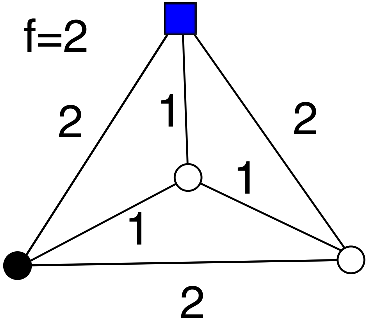

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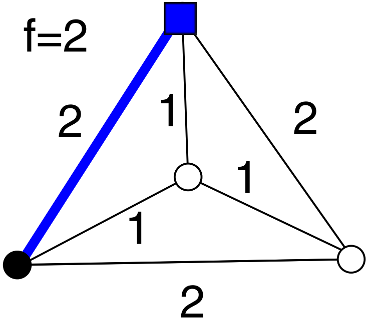

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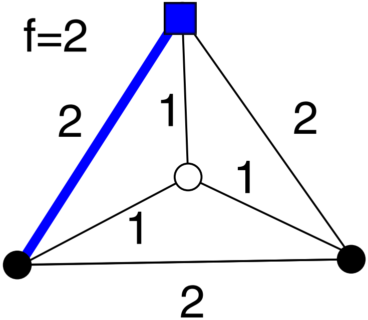
Total cost $= 2 + 2$

Online Facility Location Problem



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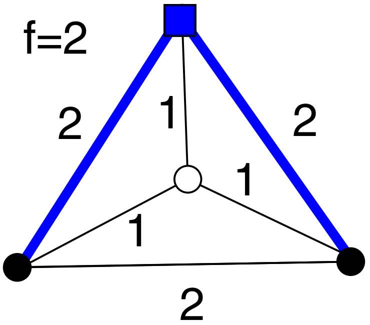

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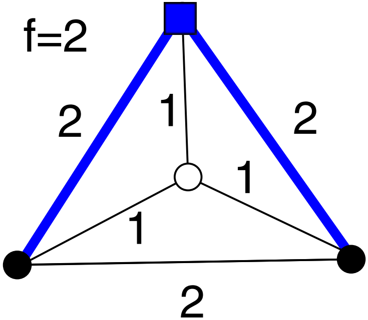

$$\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j))$$

Total cost $= 2 + 2 + 2 = 6$.

Online Facility Location LP Formulation

Online Facility Location LP Formulation

Linear programming relaxation

$$\begin{array}{lll}
\min & \sum_{i\in F} f(i)y_i + \sum_{j\in D}\sum_{i\in F} d(j,i)x_{ji} & \\
\text{s.t.} & x_{ji} \leq y_i & \text{for } j \in D \text{ and } i \in F, \\
& \sum_{i\in F} x_{ji} \geq 1 & \text{for } j \in D, \\
& y_i \geq 0, x_{ji} \geq 0 & \text{for } j \in D \text{ and } i \in F,
\end{array}$$

Online Facility Location LP Formulation

Linear programming relaxation

$$\begin{array}{lll} \min & \sum_{i\in F} f(i)y_i + \sum_{j\in D}\sum_{i\in F} d(j,i)x_{ji} & \\ \text{s.t.} & x_{ji} \le y_i & \text{for } j \in D \text{ and } i \in F, \\ & \sum_{i\in F} x_{ji} \ge 1 & \text{for } j \in D, \\ & y_i \ge 0, x_{ji} \ge 0 & \text{for } j \in D \text{ and } i \in F, \end{array}$$

and its dual

$$\begin{array}{lll} \max & \sum_{j\in D} \alpha_j & \\ \text{s.t.} & \sum_{j\in D}(\alpha_j - d(j,i))^+ \le f(i) & \text{for } i \in F, \\ & \alpha_j \ge 0 & \text{for } j \in D. \end{array}$$

Online Facility Location Algorithm

Algorithm 5: OFL Algorithm.

Input: (G, d, f, F)
$F^a \leftarrow \emptyset$; $D \leftarrow \emptyset$;
while *a new client j' arrives* **do**
  increase $\alpha_{j'}$ until one of the following happens:
  (a) $\alpha_{j'} = d(j', i)$ for some $i \in F^a$; /* connect only */
  (b) $f(i) = (\alpha_{j'} - d(j', i)) + \sum_{j \in D}(d(j, F^a) - d(j, i))^+$ for some $i \in F \setminus F^a$; /* open and connect */
  $F^a \leftarrow F^a \cup \{i\}$; $D \leftarrow D \cup \{j'\}$; $a(j') \leftarrow i$;
end
return (F^a, a);

Online Facility Location Results

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Online Facility Location Results

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[Nagarajan and Williamson 2013] give a dual-fitting analysis for the algorithm by [Fotakis 2007].

Acknowledgements

Thank you!

Questions?