The Online Multicommodity Connected Facility Location Problem

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October 20th, 2017

Main Goals

Define and present a competitive algorithm for the
Online Multicommodity Connected Facility Location problem.

But first . . .

Combinatorial Optimization Problems

Maximization or minimization problems.

Algorithm receives an input.

Returns a solution with a cost.

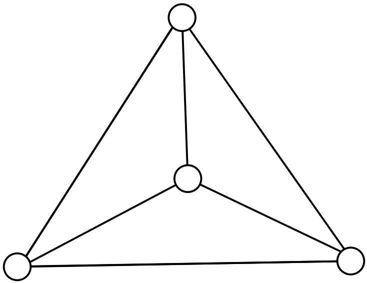
Some minimization problems are:

- Facility Location problem,
- Steiner Tree problem,
- Connected Facility Location problem.

These problems are NP-hard with constant factor approximation algorithms known.

The Facility Location Problem

The Facility Location Problem



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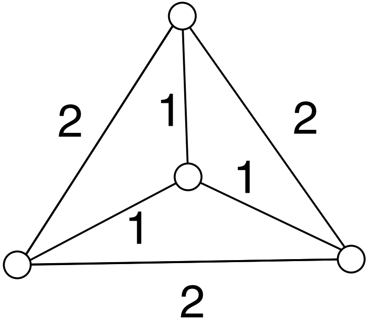

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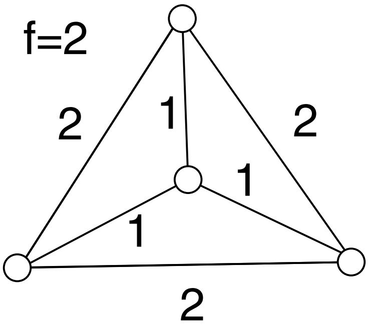

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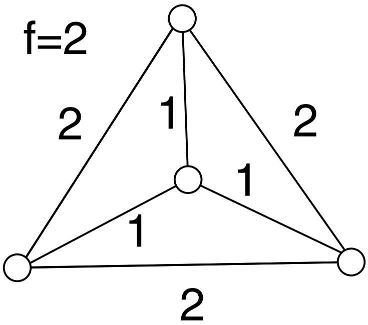
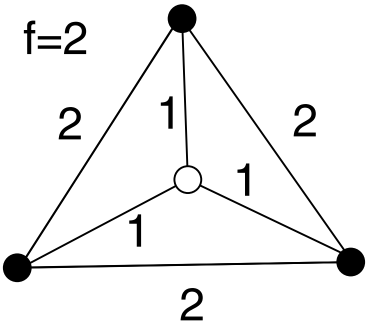


$$\min \sum_{i \in F} f(i) + \sum_{j \in D} d(j, F)$$

The Facility Location Problem



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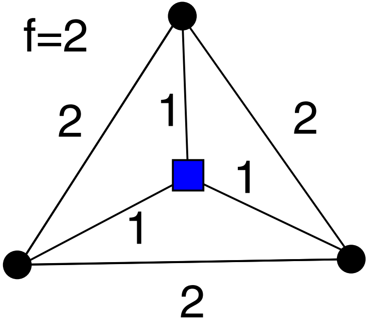
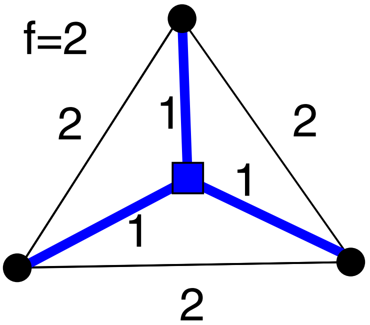
The Facility Location Problem



$$\min \sum_{i \in F} f(i) + \sum_{j \in D} d(j, F)$$

Total cost $= 2$

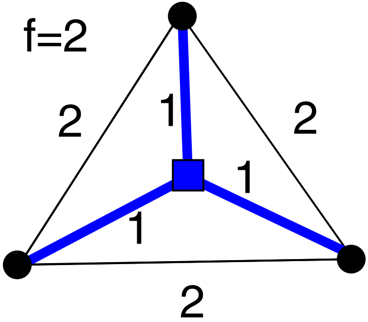
The Facility Location Problem



$$\min \sum_{i \in F} f(i) + \sum_{j \in D} d(j, F)$$

Total cost $= 2 \ + 3$

The Facility Location Problem

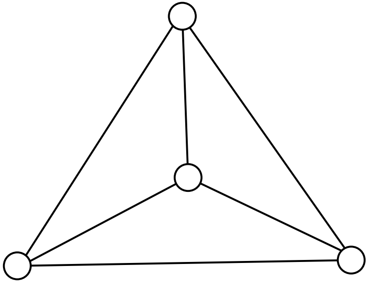


$$\min \sum_{i \in F} f(i) + \sum_{j \in D} d(j, F)$$

Total cost $= 2 + 3 = 5$

The Steiner Tree Problem

The Steiner Tree Problem



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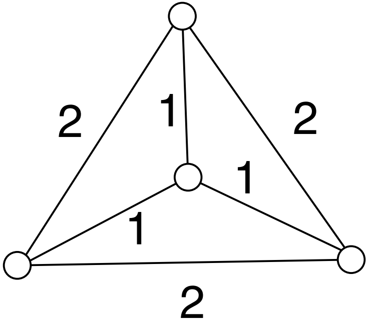

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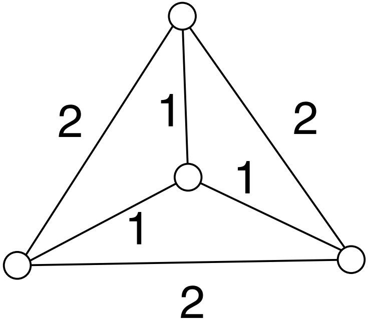
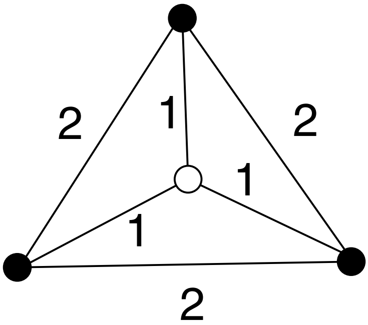


$$\min \sum_{e \in T} d(e)$$

The Steiner Tree Problem



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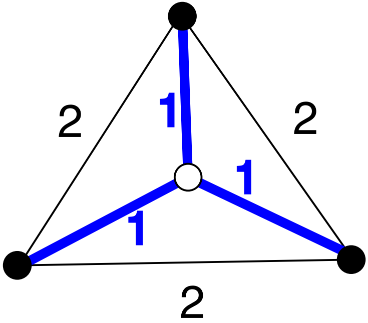
The Steiner Tree Problem



$$\min \sum_{e \in T} d(e)$$

Total cost $= 3$

The Connected Facility Location Problem

Combination of the Facility Location and the Steiner Tree problems through the rent-or-buy model.

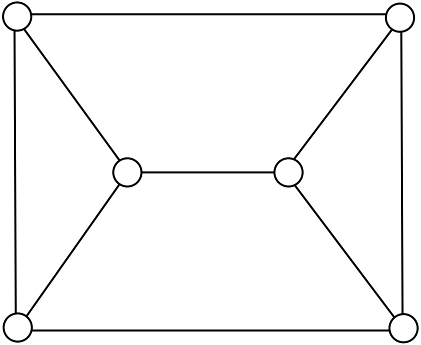
Motivation is to build a two-layer network.

Algorithm receives a set of clients and connects each client to an opened facility.

Also, it builds an expensive tree which connects all facilities.

Connected Facility Location Problem

Connected Facility Location Problem



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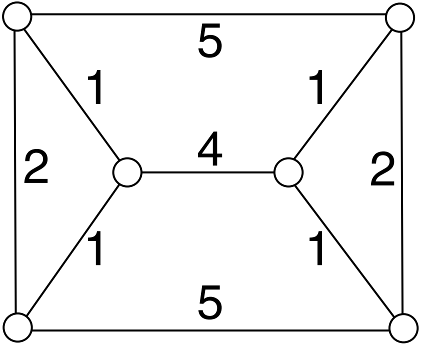

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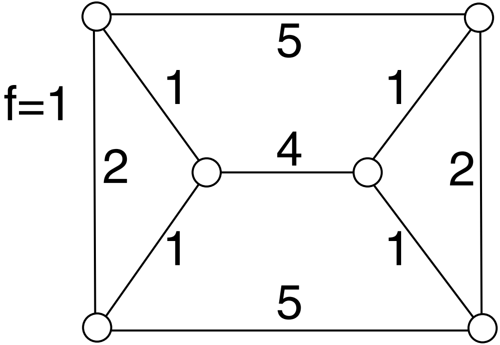

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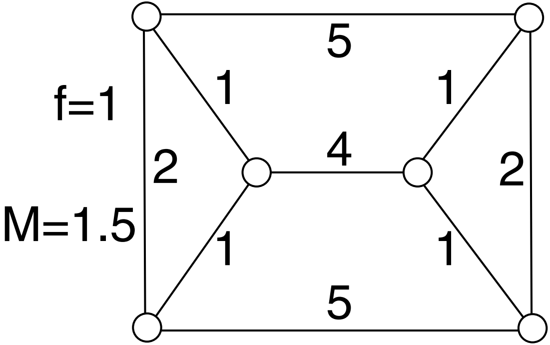

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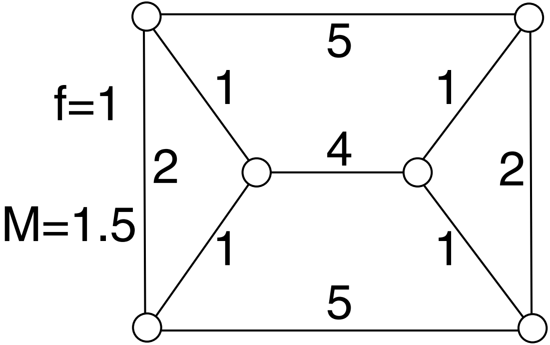
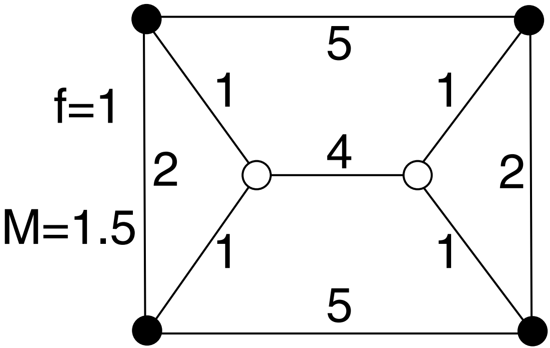


$$\min \sum_{i \in F} f(i) + \sum_{j \in D} d(j, F) + M \sum_{e \in T} d(e)$$

Connected Facility Location Problem



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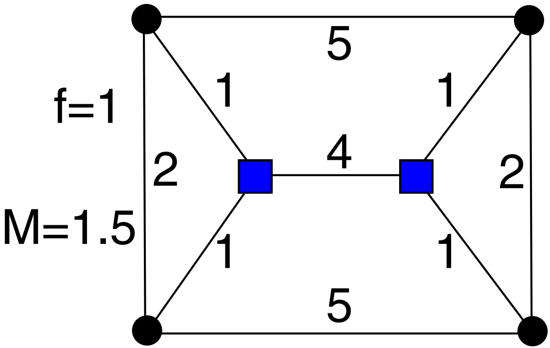
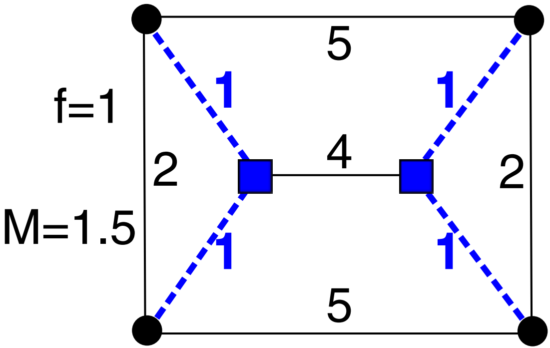
Connected Facility Location Problem



$$\min \sum_{i \in F} f(i) + \sum_{j \in D} d(j,F) + M \sum_{e \in T} d(e)$$

Total cost $= 2$

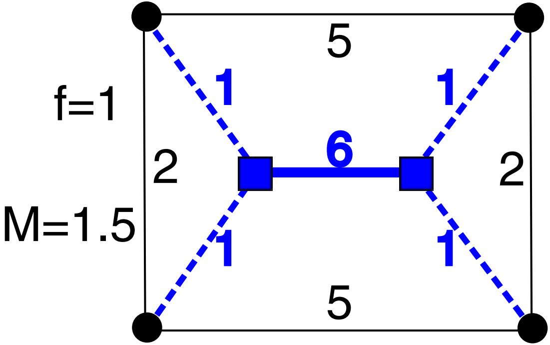
Connected Facility Location Problem



$$\min \sum_{i \in F} f(i) + \sum_{j \in D} d(j, F) + M \sum_{e \in T} d(e)$$

Total cost $= 2 \ + 4$

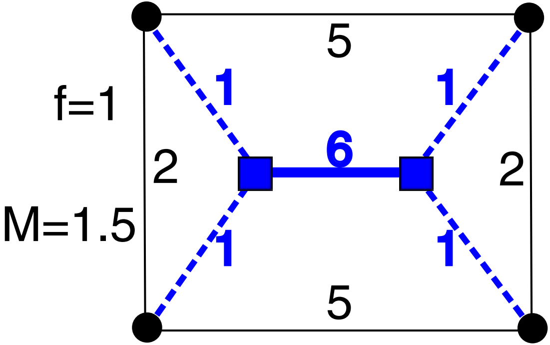
Connected Facility Location Problem



$$\min \sum_{i \in F} f(i) + \sum_{j \in D} d(j, F) + M \sum_{e \in T} d(e)$$

Total cost $= 2 \ + 4 \ + 6 =$

Connected Facility Location Problem



$$\min \sum_{i \in F} f(i) + \sum_{j \in D} d(j, F) + M \sum_{e \in T} d(e)$$

Total cost $= 2 \ + 4 \ + 6 = \ 12$

Multicommodity Connected Facility Location

Generalization of the Connected Facility Location problem.

Proposed by Fabrizio Grandoni and Thomas Rothvoß, who presented a constant approximation for it.

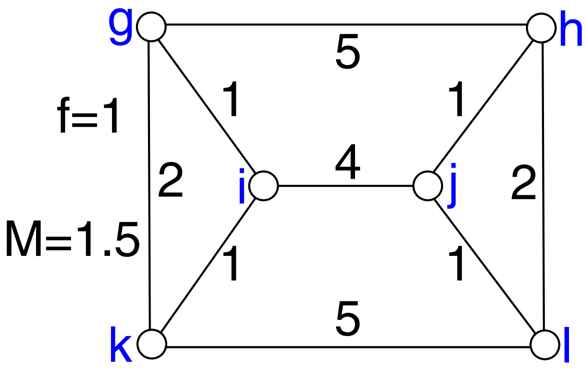
Algorithm receives a set of pairs to connect.

It may rent or buy edges and open facilities to connect each pair.

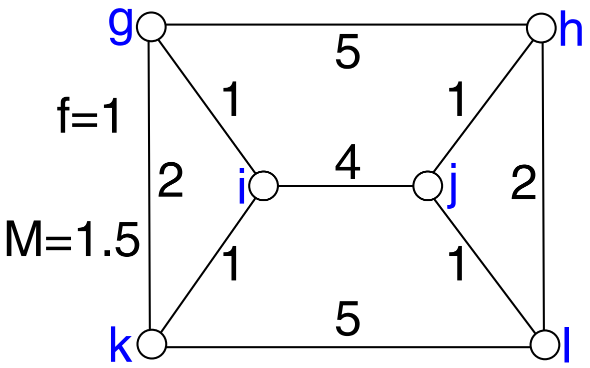
The path connecting a pair may only change between rented and bought edges at an opened facility.

Multicommodity Connected Facility Location

Multicommodity Connected Facility Location

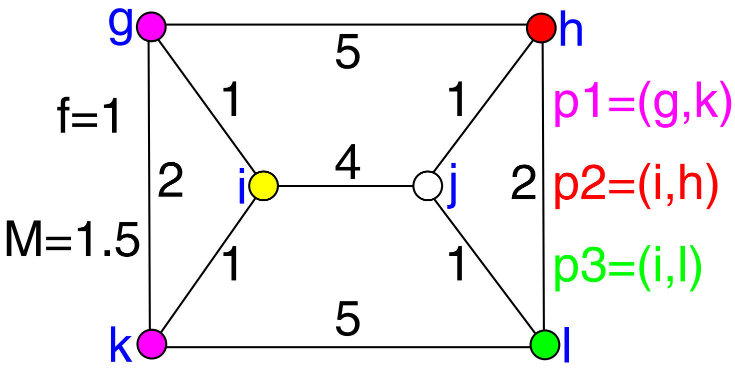


Multicommodity Connected Facility Location



$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

Multicommodity Connected Facility Location



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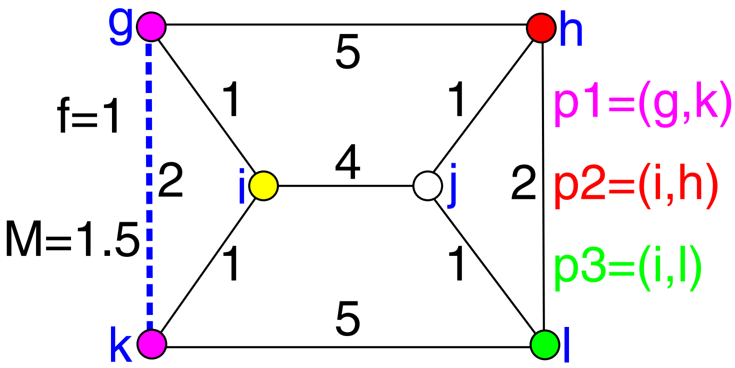
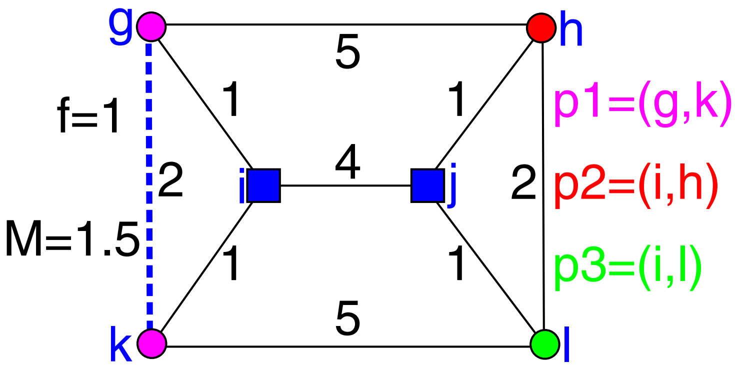
Multicommodity Connected Facility Location



$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

Total cost $= 2$

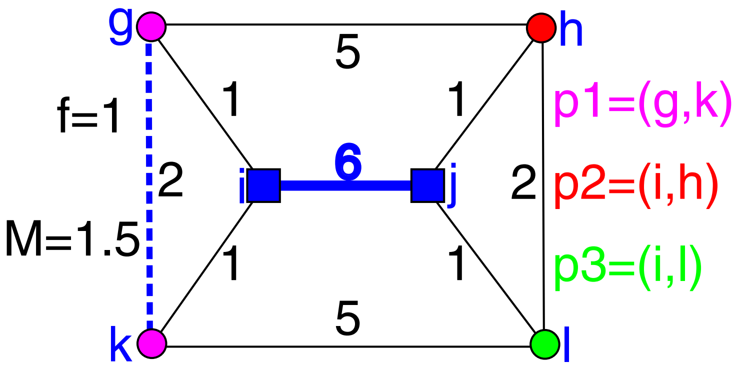
Multicommodity Connected Facility Location



$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

Total cost $= 2 \ + 2$

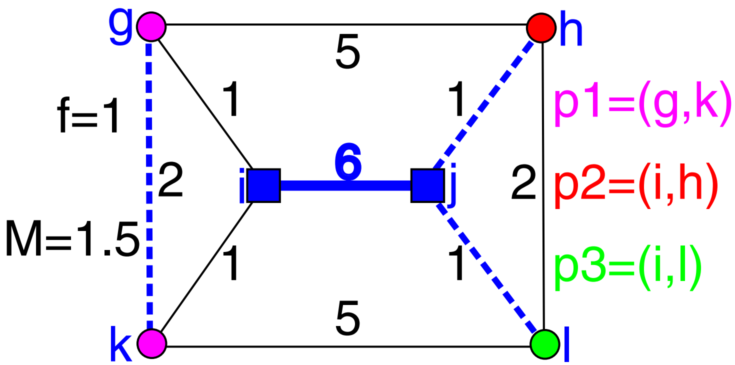
Multicommodity Connected Facility Location



$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

Total cost $= 2 \; + 2 \; + 6$

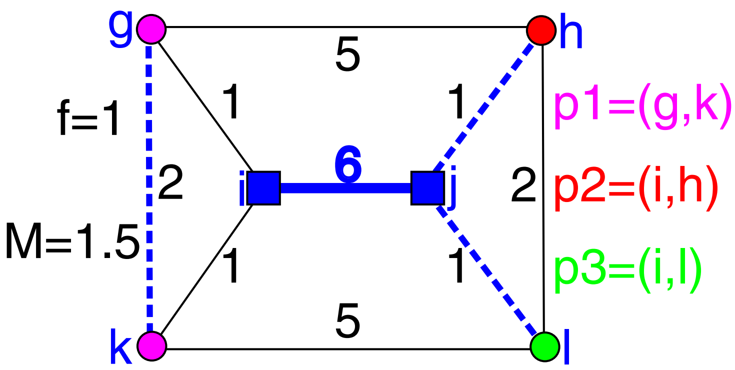
Multicommodity Connected Facility Location



$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

Total cost $= 2 \ + 2 \ + 6 \ + 2 =$

Multicommodity Connected Facility Location



$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

Total cost $= 2 \; + 2 \; + 6 \; + 2 = \; 12$

Online Problems and Competitive Analysis

Parts of the input are revealed one at a time.

Each part is served before the next one arrives.

No decision made may be changed in the future.

An online algorithm ALG is c-competitive if:

$$\mathrm{ALG}(I) \leq c\,\mathrm{OPT}(I) \ ,$$

for every input I.

Competitive ratio is similar to approximation ratio.

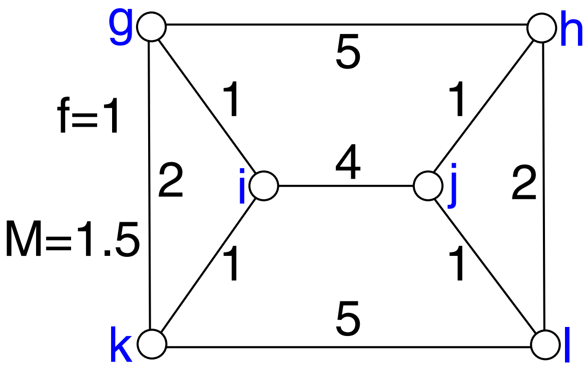
Online Multicommodity CFL Problem

Online version of the Multicommodity Connected Facility Location problem.

Pairs arrive one at a time and their nodes must be immediately connected to each other.

Opened facilities and rented or bought edges may not be removed in the future.

Online Multicommodity CFL Problem



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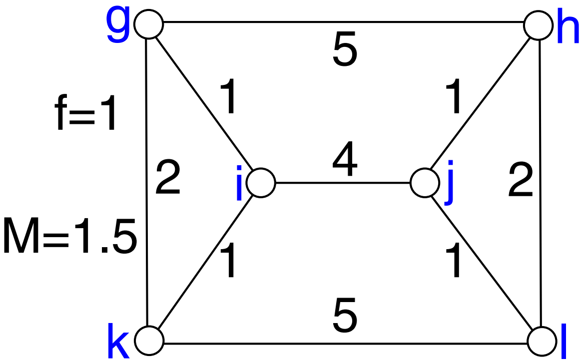

$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

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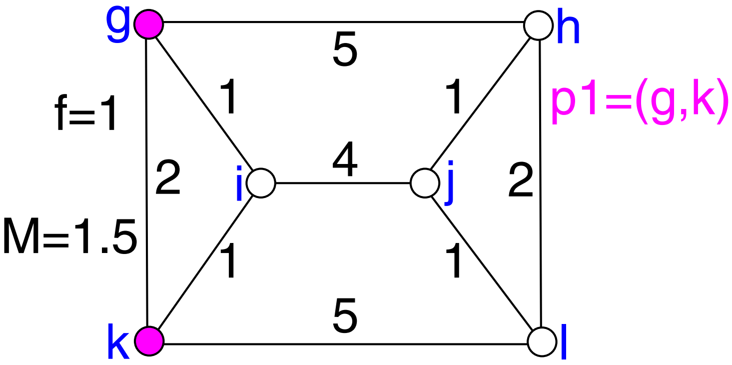

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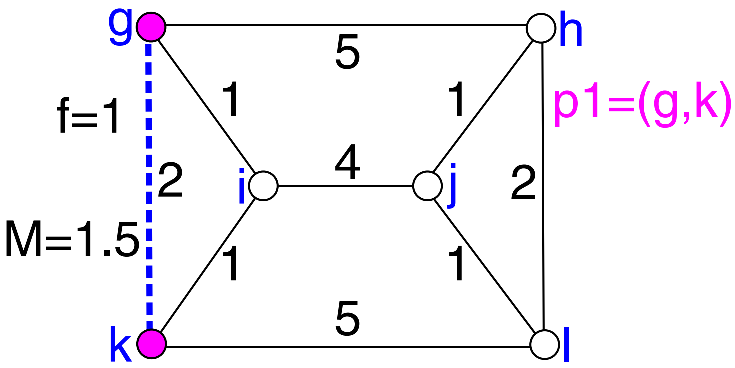
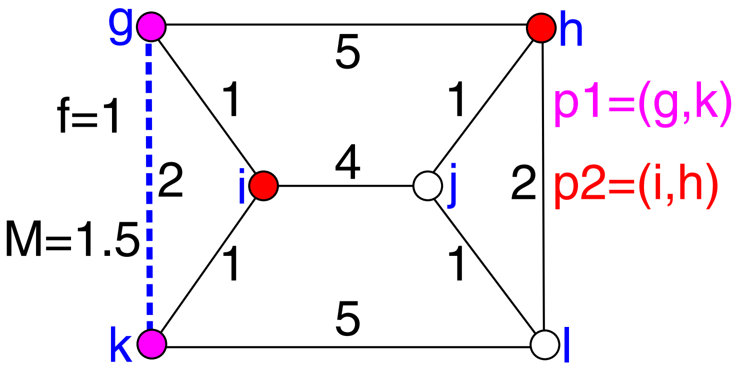


$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

Total cost $= 2$

Online Multicommodity CFL Problem



$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

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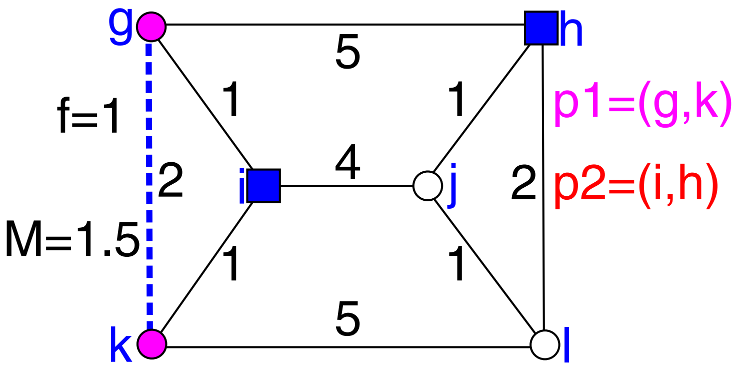
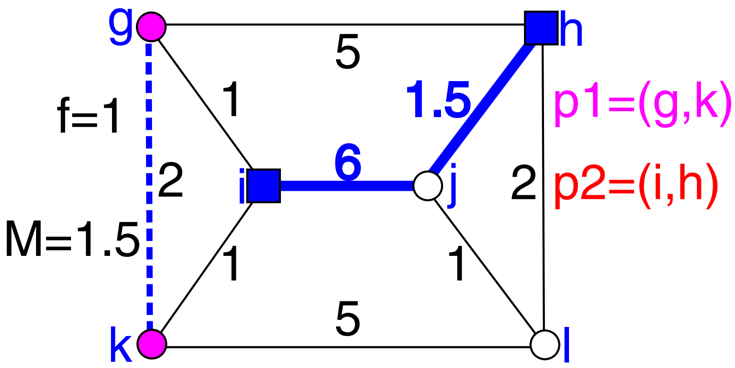


$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

Total cost $= 2 \;\; + 2$

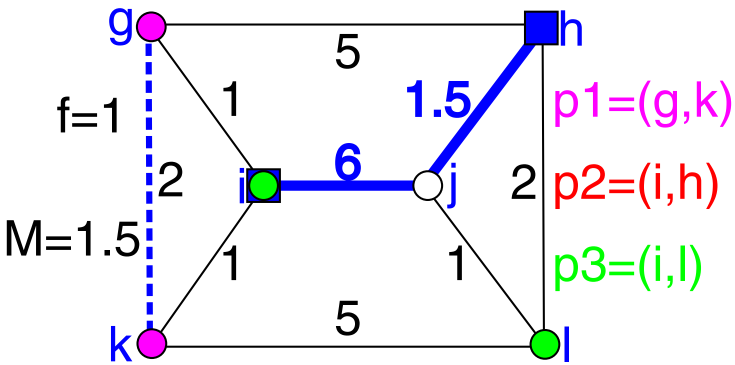
Online Multicommodity CFL Problem



$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

Total cost $= 2 \ + 2 \ + 7.5$

Online Multicommodity CFL Problem



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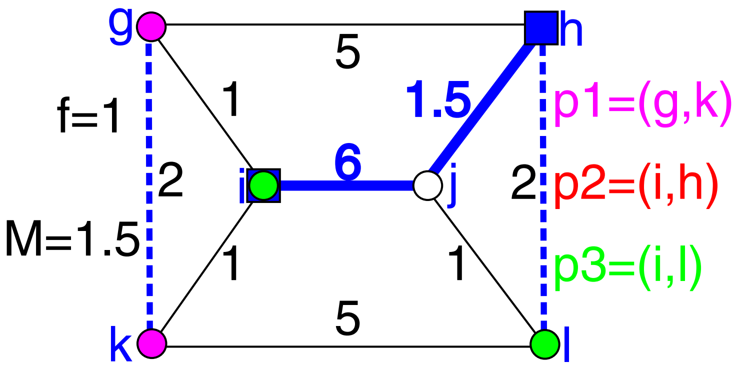
Total cost $= 2 \; + 2 \; + 7.5$



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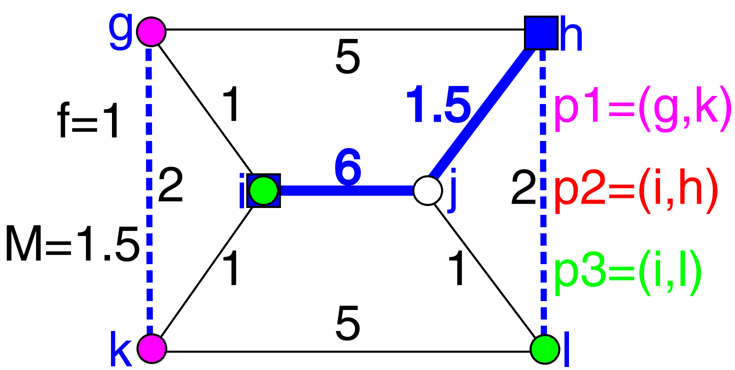
Total cost $= 2 \ + 2 \ + 7.5 \ + 2 =$

Online Multicommodity CFL Problem



$$\min \sum_{i \in F} f(i) + \sum_{p \in P} \sum_{e \in E_p^r} d(e) + M \sum_{e \in E^b} d(e)$$

Total cost $= 2 \ + 2 \ + 7.5 \ + 2 = \ 13.5$

Online Multicommodity CFL Algorithm

We present a sample-and-augment algorithm inspired on the algorithm for MCFL due to Grandoni and Rothvoß.

Sample-and-Augment is a technique, due to Gupta et al., to design randomized algorithms for rent-or-buy problems.

We highlight that the Online Multicommodity Connected Facility Location problem is not a typical rent-or-buy problem.

Because the constraints on rented edges are distinct from those on bought edges.

However, it still has a cost scaling factor which justify the use of this technique.

Algorithm 1: Draft of Algorithm for the OMCFL problem.

Input: (G, d, f, M)

while *a new pair* $p = (s, t)$ *arrives* **do**

  decide if and which facilities to open when serving s and t;

  ▷ algorithm for the Online Prize-Collecting Facility Location

  mark p with probability $\frac{1}{M}$; ▷ balance cost scaling factor

  if p *is marked* **then**

    open facilities to which s and t are assigned and update F^a;

    choose edges to connect these facilities and update E^b;

    ▷ algorithm for the Online Steiner Forest

    add zero cost edges connecting opened facilities which are in the same bought component;

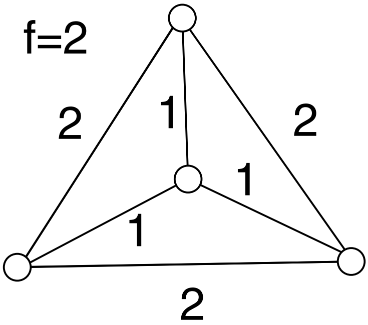
  consider an (s, t)-shortest path in G;

  let E_p^r be the non zero cost edges of this path;

return $(F^a, E^b, (E_p^r)_{p \in P})$;

Online Prize-Collecting Facility Location Problem

Online Prize-Collecting Facility Location Problem



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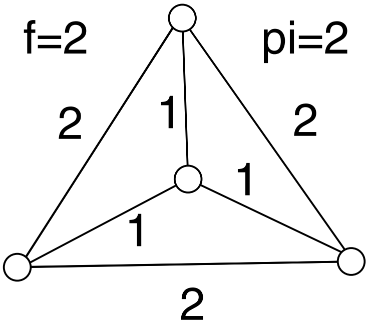

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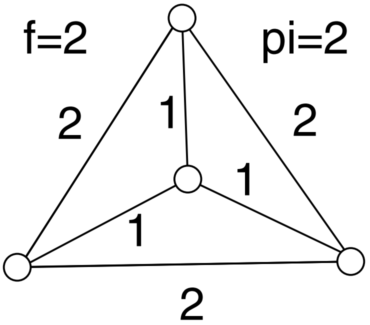
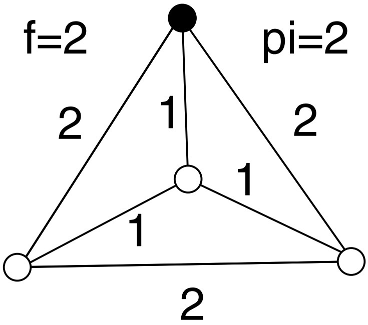


$$\min \sum_{i \in F^a} f(i) + \sum_{j \notin D^\phi} d(j, \phi(j)) + \sum_{j \in D^\phi} \pi(j)$$

Online Prize-Collecting Facility Location Problem



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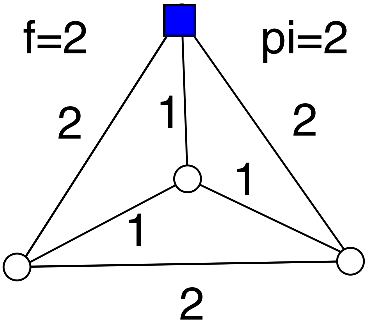
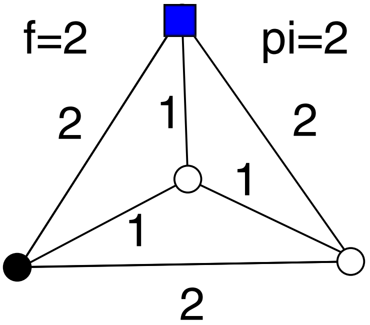


$$\min \sum_{i \in F^a} f(i) + \sum_{j \notin D^\phi} d(j, \phi(j)) + \sum_{j \in D^\phi} \pi(j)$$

Total cost $= 2$

Online Prize-Collecting Facility Location Problem



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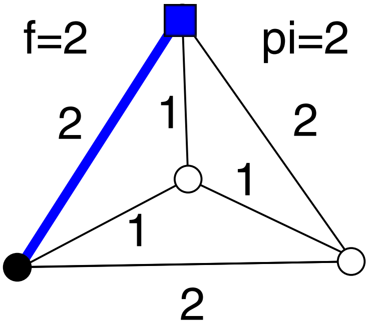
Total cost $= 2$

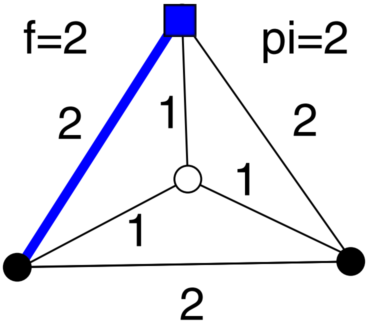
Online Prize-Collecting Facility Location Problem



$$\min \sum_{i \in F^a} f(i) + \sum_{j \notin D^\phi} d(j, \phi(j)) + \sum_{j \in D^\phi} \pi(j)$$

Total cost $= 2 + 2$

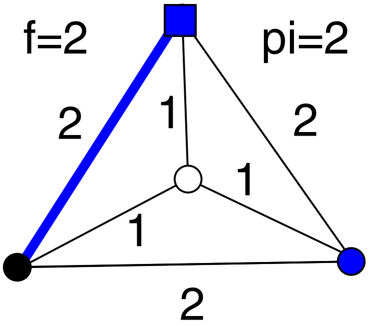
Online Prize-Collecting Facility Location Problem



$$\min \sum_{i \in F^a} f(i) + \sum_{j \notin D^\phi} d(j, \phi(j)) + \sum_{j \in D^\phi} \pi(j)$$

Total cost $= 2 + 2$

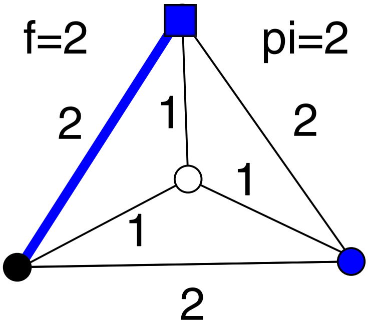
Online Prize-Collecting Facility Location Problem



$$\min \sum_{i \in F^a} f(i) + \sum_{j \notin D^\phi} d(j, \phi(j)) + \sum_{j \in D^\phi} \pi(j)$$

$$\text{Total cost} = 2 + 2 + 2$$

Online Prize-Collecting Facility Location Problem



$$\min \sum_{i \in F^a} f(i) + \sum_{j \notin D^\phi} d(j, \phi(j)) + \sum_{j \in D^\phi} \pi(j)$$

$$\text{Total cost} = 2 + 2 + 2 = 6$$

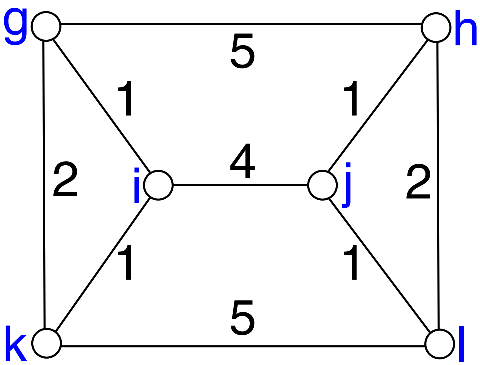
Online Prize-Collecting Facility Location Problem

Elmachtoub and Levi, and San Felice et al. independently presented $O(\log n)$-competitive algorithms for the OPFL.

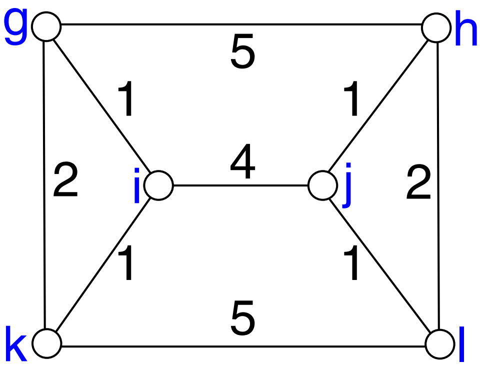
Since the OPFL is a generalization of the Online Facility Location problem, the $\Omega\left(\frac{\log n}{\log \log n}\right)$ lower bound due to Fotakis applies to it.

Online Steiner Forest Problem

Online Steiner Forest Problem

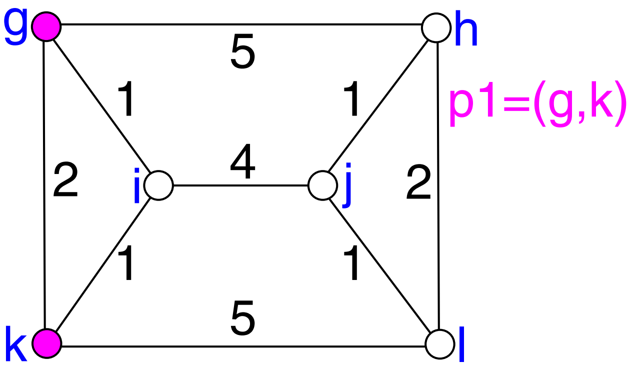


Online Steiner Forest Problem



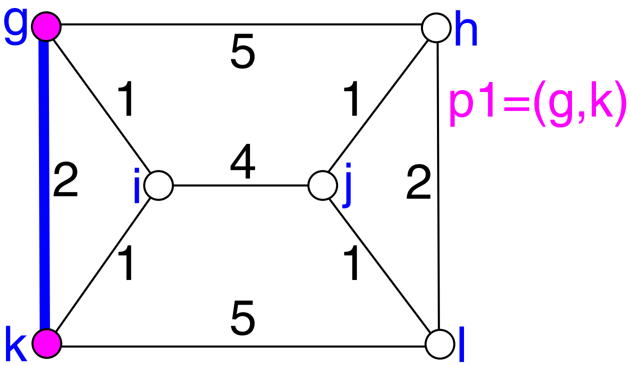
$$\min \sum_{e \in T} d(e)$$

Online Steiner Forest Problem



$$\min \sum_{e \in T} d(e)$$

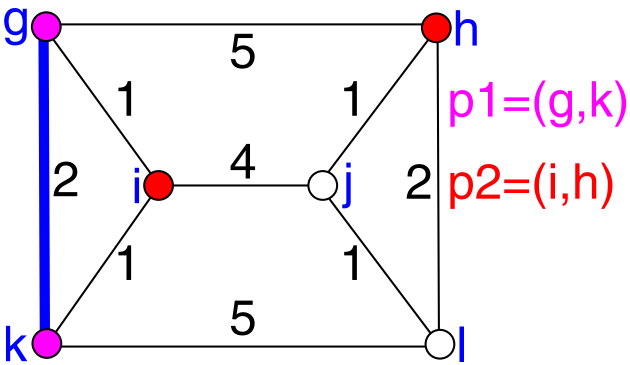
Online Steiner Forest Problem



$$\min \sum_{e \in T} d(e)$$

Total cost = 2

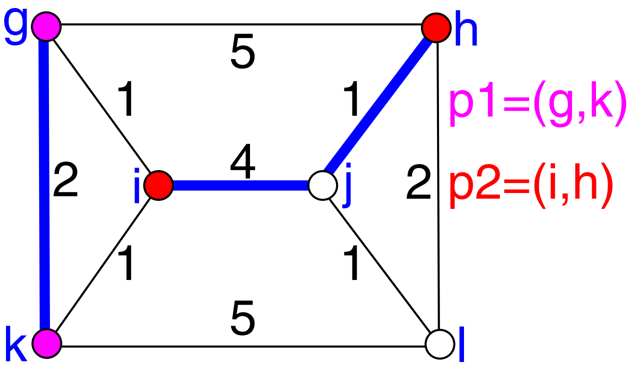
Online Steiner Forest Problem



$$\min \sum_{e \in T} d(e)$$

Total cost $= 2$

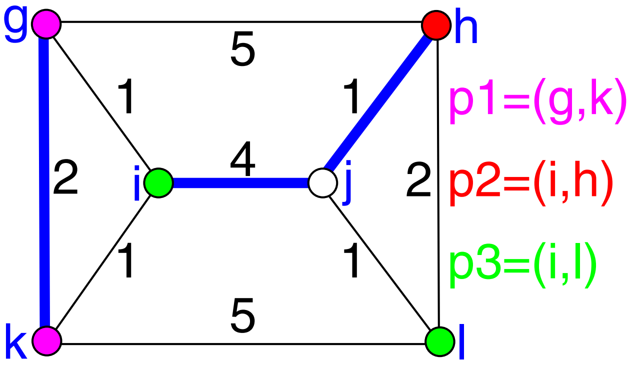
Online Steiner Forest Problem



$$\min \sum_{e \in T} d(e)$$

Total cost $= 2 \ + 5$

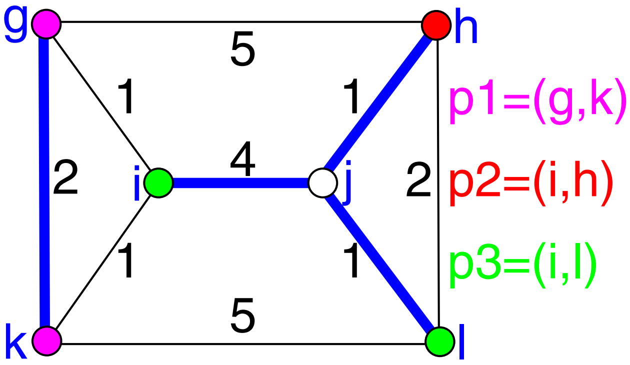
Online Steiner Forest Problem



$$\min \sum_{e \in T} d(e)$$

Total cost $= 2 \; + 5$

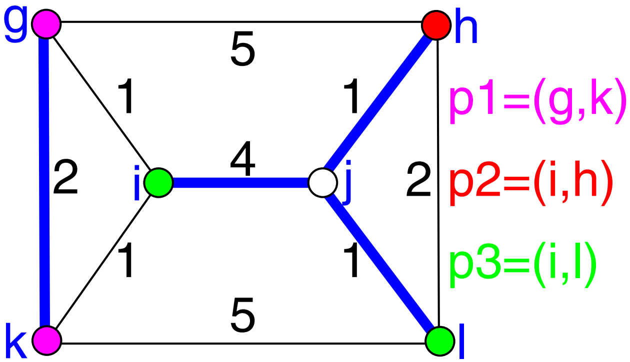
Online Steiner Forest Problem



$$\min \sum_{e \in T} d(e)$$

Total cost $= 2 \ + 5 \ + 1$

Online Steiner Forest Problem



$$\min \sum_{e \in T} d(e)$$

Total cost $= 2 + 5 + 1 = 8$

Online Steiner Forest Problem

Berman and Coulston presented a deterministic $\mathrm{O}(\log n)$-competitive algorithm for the OSF.

Also, a $\Omega(\log n)$ lower bound to the OST due to Imase and Waxman applies to the OSF.

Algorithm 2: Algorithm for the OMCFL problem.

Input: (G, d, f, M)

while *a new pair* $p = (s, t)$ *arrives* **do**

$\pi_p \leftarrow \mathrm{dist}(G, d', s, t)/2$; ▷ decide if and which facilities

send (s, π_p) and (t, π_p) to $\mathrm{ALG}_{\mathrm{OPFL}}$ obtaining $\phi(s)$ and $\phi(t)$;

if $\phi(s) \neq \mathrm{null}$ *and* $\phi(t) \neq \mathrm{null}$ **then**

mark p with probability $1/M$; ▷ balance cost scaling factor

if *p is marked* **then**

send $(\phi(s), \phi(t))$ to $\mathrm{ALG}_{\mathrm{OSF}}$ obtaining an edge set E_p^b;

$F^a \leftarrow F^a \cup \{\phi(s), \phi(t)\}$; $E^b \leftarrow E^b \cup E_p^b$;

for $x, y \in F^a$ *in the same component of* $G[E^b]$ **do**

$d'(x, y) \leftarrow 0$; $E' \leftarrow E' \cup \{xy\}$;

consider an (s, t)-shortest path in G with costs d';

let E_p^r be the edges of this path except for those in E';

return $(F^a, E^b, (E_p^r)_{p \in P})$;

Analysis of the OMCFL Algorithm

Cost of Algorithm for OMCFL is divided between facilities opening cost (O), edges buying cost (B) and edges renting cost (R):

$$\mathrm{ALG}_{\mathrm{OMCFL}}(P) = O(P) + B(P) + R(P) \ .$$

And the edges renting cost (R) is divided according to the pairs in P^{π}, P^{m} and P^{u}:

$$R(P) = R^{\pi}(P) + R^{m}(P) + R^{u}(P) \ .$$

The cost of the offline optimal solution is also divided in this way:

$$OPT_{MCFL}(P) = O^{*}(P) + B^{*}(P) + R^{*}(P) \ .$$

First an Auxiliary Lemma

Lemma

$\mathrm{OPT}_{\mathrm{PFL}}(D) \leq \mathrm{OPT}_{\mathrm{MCFL}}(P)$.

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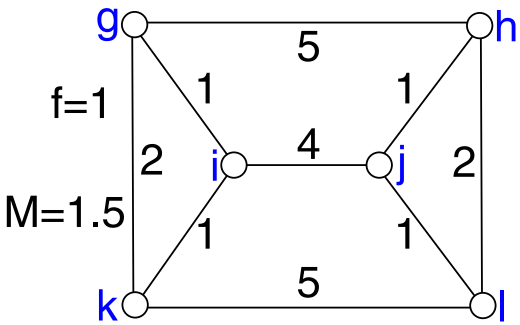

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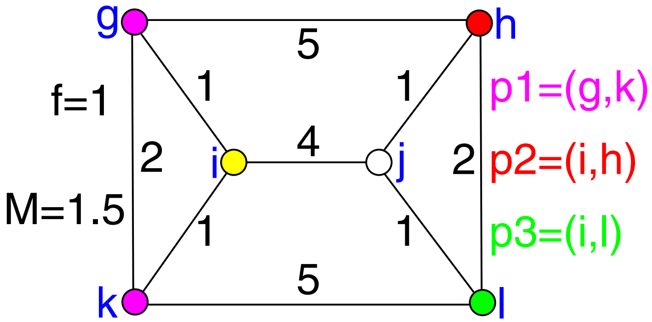

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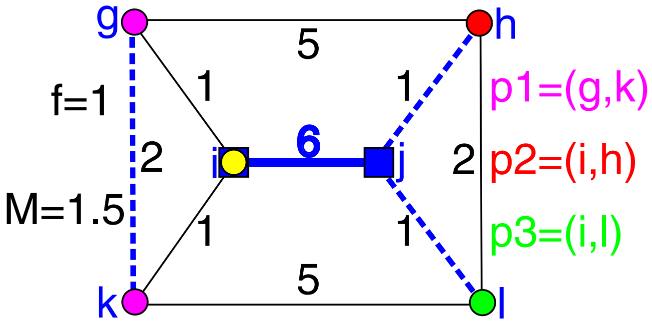

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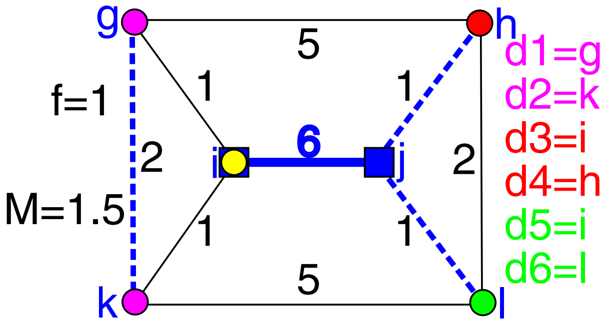

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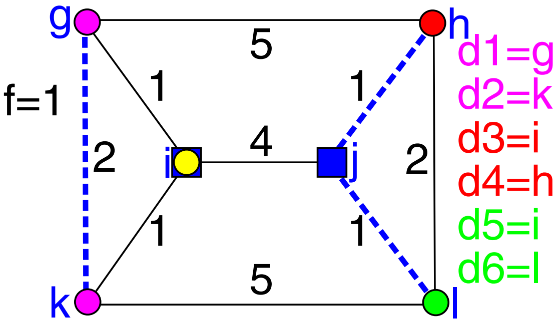

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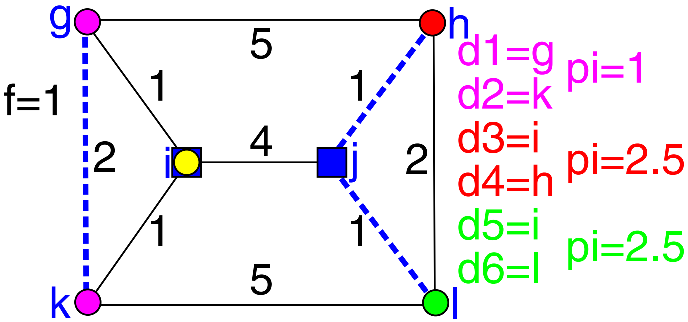

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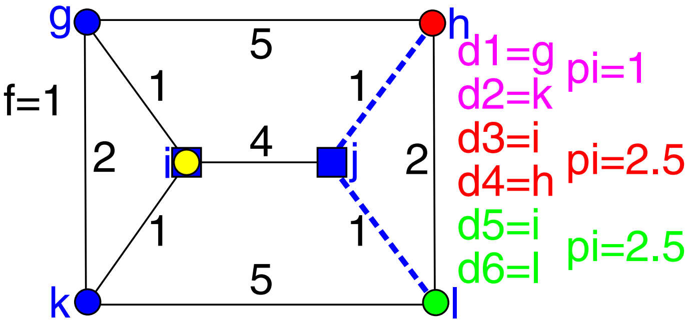
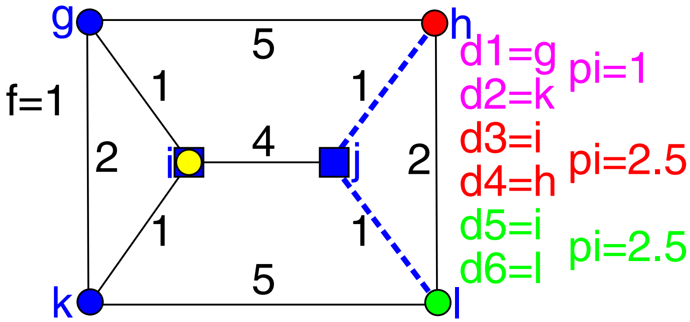


First an Auxiliary Lemma

Lemma

$\mathrm{OPT}_{\mathrm{PFL}}(D) \leq \mathrm{OPT}_{\mathrm{MCFL}}(P)$.



$$\mathrm{ALG}_{\mathrm{OPFL}}(D) \leq \mathrm{O}(\log n)\mathrm{OPT}_{\mathrm{PFL}}(P) \leq \mathrm{O}(\log n)\mathrm{OPT}_{\mathrm{MCFL}}(P) \ .$$

Some Simple Lemmas

Cost of Algorithm for OPFL is divided between facilities opening cost (O'), clients penalty cost (Π) and clients connection cost (C'):

$$\mathrm{ALG}_{\mathrm{OPFL}}(D) = O'(D) + \Pi(D) + C'(D) \ .$$

Lemma (Facility Opening Cost)

$O(P) \leq O'(D)$. *$\mathrm{ALG}_{\mathrm{OMCFL}}$ opens a subset of $\mathrm{ALG}_{\mathrm{OPFL}}$ facilities.*

Lemma (Close Pairs Renting Cost)

$R^{\pi}(P) \leq 2\Pi(D)$. *At least one node of each pair paid penalty.*

Lemma (Marked Pairs Renting Cost)

$R^{m}(P) \leq C'(D)$. *For every marked pair, its renting edges correspond to its nodes connections.*

Central Lemma

Lemma (Buying Cost)

$\mathbf{E}[B(P)] = \mathrm{O}(\log^2 n)\,\mathrm{OPT}_{\mathrm{MCFL}}(P).$

$$B(P) \leq M\,\mathrm{ALG}_{\mathrm{OSF}}(Q) = M\,\mathrm{O}(\log n)\,\mathrm{OPT}_{\mathrm{SF}}(Q) \quad .$$

$$\mathbf{E}[\mathrm{OPT}_{\mathrm{SF}}(Q)] \leq (B^*(P) + R^*(P) + C'(D))\,/\,M \quad .$$

$$\begin{aligned}\mathbf{E}[B(P)] &= \mathrm{O}(\log n)\,(B^*(P) + R^*(P) + C'(D)) \\ &= \mathrm{O}(\log n)\ (B^*(P) + R^*(P) + \mathrm{ALG}_{\mathrm{OPFL}}(D)) \\ &= \mathrm{O}(\log n)\ (\mathrm{OPT}_{\mathrm{MCFL}}(P) + \mathrm{O}(\log n)\mathrm{OPT}_{\mathrm{PFL}}(D)) \\ &= \mathrm{O}(\log^2 n)\,\mathrm{OPT}_{\mathrm{MCFL}}(P) \quad .\end{aligned}$$

Final Lemma

Lemma (Unmarked Pairs Renting Cost)

$\mathbf{E}[R^u(P)] = \mathrm{O}(\log^2 n)\,\mathrm{OPT}_{\mathrm{MCFL}}(P)$.

$S_p = E_p^u$ if $p \notin P^m$ and $Z_p = E_p^b$ if $p \in P^m$.

$$\mathbf{E}\left[\sum_{e \in E_p^u} d(e) \mid d(S_p), d(Z_p)\right] = \frac{M-1}{M} d(S_p) \leq d(S_p) \ ,$$

$$\mathbf{E}\left[\sum_{e \in E_p^b} M d(e) \mid d(S_p), d(Z_p)\right] = \frac{1}{M} M\, d(Z_p) = d(Z_p) \ .$$

$$\mathbf{E}\left[\sum_{e \in E_p^u} d(e)\right] \leq \mathbf{E}\left[\sum_{e \in E_p^b} M d(e)\right] + d(s, \phi(s)) + d(t, \phi(t)) \ .$$

$$\mathbf{E}[R^u(P)] \leq \mathbf{E}[B(P)] + C'(D) = \mathrm{O}(\log^2 n)\,\mathrm{OPT}_{\mathrm{MCFL}}(P) \ .$$

Main Result

Theorem

$$\mathbf{E}[\mathrm{ALG}_{\mathrm{OMCFL}}(P)] = \mathrm{O}(\log^2 n)\,\mathrm{OPT}_{\mathrm{MCFL}}(P).$$

$$\begin{aligned}
\mathbf{E}[\mathrm{ALG}_{\mathrm{OMCFL}}(P)] &= \mathbf{E}[O(P)] + \mathbf{E}[B(P)] + \mathbf{E}[R(P)] \\
&= \mathbf{E}[O(P)] + \mathbf{E}[B(P)] \\
&\quad + \mathbf{E}[R^{\pi}(P) + R^{m}(P) + R^{u}(P)] \\
&\leq O'(D) + \mathrm{O}(\log^2 n)\,\mathrm{OPT}_{\mathrm{MCFL}}(P) \\
&\quad + 2\Pi(D) + C'(D) + \mathrm{O}(\log^2 n)\,\mathrm{OPT}_{\mathrm{MCFL}}(P) \\
&= \mathrm{O}(\log^2 n)\,\mathrm{OPT}_{\mathrm{MCFL}}(P) \enspace .
\end{aligned}$$

Final Remarks

With a small change in the algorithm we are able to achieve a logarithmic bound on the expected buying cost ($B(P)$). Thus, we have:

Theorem

In the special case of OMCFL in which $M = 1$, we have

$$\mathrm{ALG2}_{\mathrm{OMCFL}}(P) = \mathrm{O}(\log n)\,\mathrm{OPT}_{\mathrm{MCFL}}(P) \ .$$

However, we are still working to improve the bound on the expected renting cost of unmarked clients ($R^u(P)$).

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Acknowledgements

That's all!

Questions?